Characteristics of Dual √3 subdivision schemes

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Dual √3 subdivision schemes are strange beasts. They live somewhere between the useful subdivision schemes and the unusable subdivision schemes. Studying such schemes can provide insights into the behaviour of the more useful schemes.

On this page we compare Dual √3 against Primal √3, Primal √2 and Dual √2 schemes.

On the other page we analyse the characteristics of Dual √3 schemes. This shows that, despite some superficially nice features, Dual √3 schemes have serious drawbacks which make them unusable in practice.

Terminology

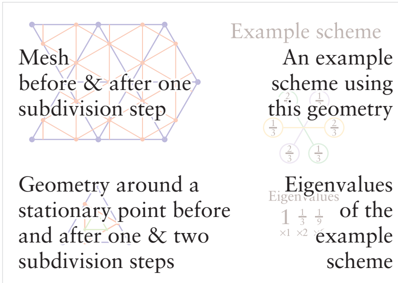
Primal: both vertices and faces centres map to vertices

Dual: both vertices and faces centres map to face centres

√3: a triangular scheme, two subdivision steps produce a ternary scheme

√2: a quadrilateral scheme, two subdivision steps produce a binary scheme

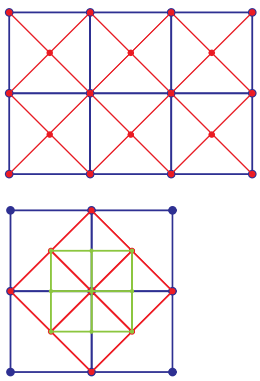
Key to each of the four figures below



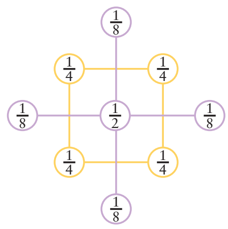
Quadrilateral √2 | *Triangular √3*

Primal (vertex → vertex, face centre → vertex)

Quadrilateral √2



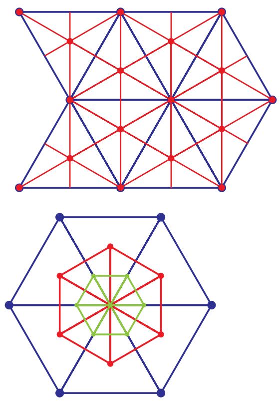
Example scheme
[Velho & Zorin, 2001]



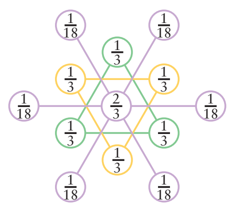
Eigenvalues

1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	0
×1	×2	×3	×4	×5	×10	×8	×1

Triangular √3



Example scheme
[Kobbelt, 2000]

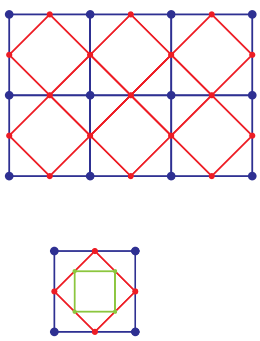


Eigenvalues

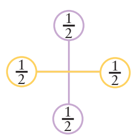
1	$\frac{1}{3}$	$\frac{1}{9}$	0
×1	×2	×3	×1

Dual (vertex → face centre, face centre → face centre)

Quadrilateral √2



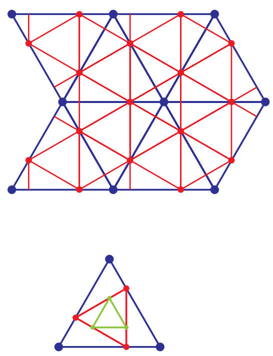
Example scheme
[Peters & Reif, 1997]



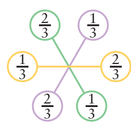
Eigenvalues

1	$\frac{1}{2}$	$\frac{1}{4}$	0
×1	×2	×8	×1

Triangular √3



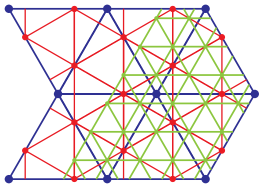
Example scheme
[2002]



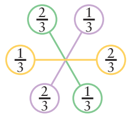
Eigenvalues

1	$\frac{1}{3}$	$\frac{1}{9}$
×1	×2	×6

The basic scheme

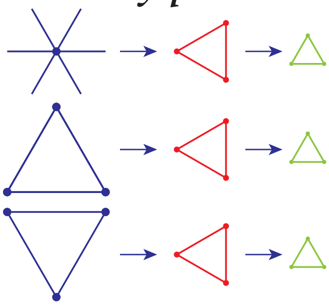


The geometry of the Dual √3 subdivision scheme is shown above. We see part of a base regular triangular mesh, with a first level of subdivision and part of the second level of subdivision.



The simplest Dual √3 subdivision scheme places new vertices one third of the way along each edge. This is a very small mask, allowing for efficient local computation of new vertex locations.

Stationary points



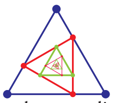
Vertices map to left triangles which map to up triangles

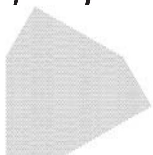
Up triangles map to left triangles which map to up triangles

Down triangles map to left triangles which map to up triangles

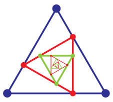
The stationary points, about which eigenanalysis can be performed, are the face centres of up triangles. The face centres of left triangles are also stationary. Vertices and the face centres of down triangles and right triangles are not stationary, but become left or up triangles after one subdivision step.

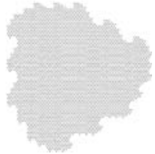
Alternatives & their footprints





rotate in alternate directions on alternate steps

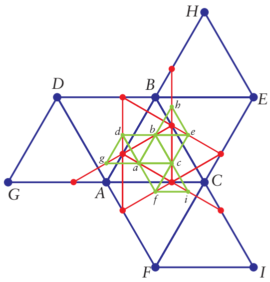




rotate in same direction on every step

A footprint shows which vertices in the subdivided mesh are affected by a given vertex in the original mesh. We show the footprints after six subdivision steps. The lower footprint is fractal (as is the Primal √3 footprint). Both footprints have only threefold rotational symmetry.

Eigenanalysis



$$\begin{bmatrix} a\\ b\\ c\\ d\\ e\\ f\\ g\\ h\\ i \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 5 & 2 & 2 & & & & & & \\ 2 & 5 & 2 & & & & & & \\ 2 & 2 & 5 & & & & & & \\ 4 & 4 & & 1 & & & & & \\ & 4 & 4 & & 1 & & & & \\ 4 & & 4 & & & 1 & & & \\ 6 & 2 & & & & & 1 & & \\ & 6 & 2 & & & & & 1 & \\ 2 & & 6 & & & & & & 1 \end{bmatrix}\begin{bmatrix} A\\ B\\ C\\ D\\ E\\ F\\ G\\ H\\ I \end{bmatrix}$$

Note the lack of reflection symmetry in the region which must be analysed.

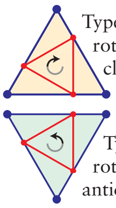
Eigenvalues

$$1 \quad \tfrac{1}{3} \quad \tfrac{1}{3} \quad \tfrac{1}{9} \quad \tfrac{1}{9} \quad \tfrac{1}{9} \quad \tfrac{1}{9} \quad \tfrac{1}{9} \quad \tfrac{1}{9}$$

The limit surface is C1 in the regular case.

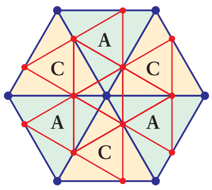
Extraordinary points

Any subdivision scheme must be able to handle extraordinary points. In this case, vertices that have valency other than 6. There appears to be no sensible way in which Dual √3 schemes can cope with such points.



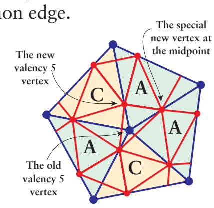
Type **C** triangle: rotate by 60° clockwise

Type **A** triangle: rotate by 60° anticlockwise

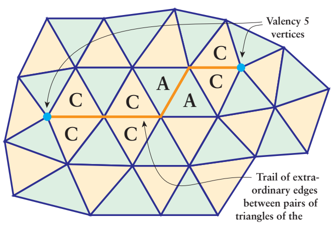


Type **A** and type **C** triangles must alternate around a vertex for the scheme to work. In a regular mesh this is easy to guarantee, with no two triangles of the same type sharing a common edge.

However, around an odd-valence vertex, it is impossible to alternate **A** and **C** triangles around the vertex. There will have to be two triangles of the same type sharing a common edge. The new vertex for this edge can be placed halfway along the edge.

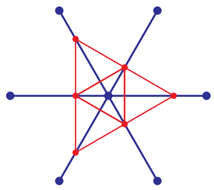


Unfortunately, an extraordinary vertex has an influence beyond its immediate neighbourhood. The inability to alternate **A** and **C** triangles propagates out to the 2-ring around the vertex, then to the 3-ring, and so on until a second extraordinary vertex intercepts the trail of pairs of triangles. Wherever two triangles of the same type share a common edge, there is a special case in the connectivity of the subdivided mesh.



Zeilfelder [2002] has shown that it is possible to label all triangles in the mesh so that nowhere are more than two triangles of the same type adjacent. This provides some limitation on the artefacts but finding an appropriate labelling is a global optimisation problem.

Breaking symmetry



A characteristic of Dual √3 schemes is that they break symmetry. Around any regular source vertex, every alternate edge will have a refined vertex one third of the way along it, while the other edges have a refined vertex two thirds of the way along the edge (as measured from the source vertex). Centres of six-fold symmetry in the source mesh thus become centres of three-fold symmetry in the new mesh. This symmetry breaking does not occur in Primal √3 schemes nor in either Primal or Dual √2 schemes.

Conclusion

Dual √3 subdivision has a very small mask, like Peters & Reif's simplest scheme, and shares the slow refinement property of all √3 and √2 schemes. However, it has serious drawbacks which make it unusable as a practical subdivision scheme. In particular: it breaks symmetry, which means that the same source mesh can have (slightly) different refinements depending on triangle labelling; and it requires a global optimisation problem to be solved at each subdivision step in order to handle extraordinary points.

References

L. Kobbelt, "√3 subdivision", *SIGGRAPH 2000*, pp. 103–112, 2000.

J. Peters & U. Reif, "The simplest subdivision scheme for smoothing polyhedra", *ACM Transactions on Graphics*, **16**(4):420–431, 1997.

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