Ternary and Three-point Univariate Subdivision Schemes

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Abstract. A family of interpolating 3-point ternary subdivision schemes is shown to exist and have C^1-continuity. A family of interpolating 4-point ternary subdivision schemes is shown to exist and have C^2-continuity. An approximating 3-point ternary scheme has been found and shown to have C^2 continuity. An approximating 3-point binary scheme is derived and shown to have C^3 continuity. The generating function formalism is used to analyze the continuity properties of these schemes. These are compared with the established schemes.

§1. Introduction

Most work in the area of subdivision schemes has considered binary schemes with an even number of control points. Following a similar argument to that used in [2], we decided to investigate schemes with an odd number of control points, specifically 3-point schemes. This led to a more general investigation of ternary subdivision schemes.

For symmetry reasons, it is obvious that an interpolating binary subdivision scheme which utilizes the closest k points, for k odd, reduces to a scheme which utilizes just the closest $k-1$ points, $k-1$ even. There is thus no 3-point interpolating binary subdivision scheme. Ternary subdivision, on the other hand, does allow for an interpolating 3-point subdivision scheme. A family of such schemes has been shown to exist and have C^1 continuity. Further investigation led to discovery of a family of interpolating 4-point ternary subdivision schemes which have C^2 continuity [6].

Investigation of approximating 3-point schemes has led to two interesting subdivision schemes. An approximating 3-point ternary scheme has been found and shown to have C^2 continuity. An approximating 3-point

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Interpolating		
Scheme	Continuity	Mask
2-point	C^0	$\frac{1}{2}[1,2,1]$
4-point	C^1	$\frac{1}{16}[-1,0,9,16,9,0,-1]$
6-point	C^2	$[\theta, 0, -3\theta - \frac{1}{16}, 0, 2\theta + \frac{9}{16}, 1, 2\theta + \frac{9}{16}, 0, -3\theta - \frac{1}{16}, 0, \theta]$
Approximating		
2-point	C^1	$\frac{1}{4}[1,3,3,1]$
3-point*	C^3	$\frac{1}{16}[1,5,10,10,5,1]$

Tab. 1. Table showing results for binary schemes, where $0 < \theta < 0.02$.
* Indicates schemes presented and analyzed in this paper.

binary scheme, which uses corner-cutting similar in spirit to the 2-point scheme $\frac{1}{4}[1,3,3,1]$, can be derived and shown to have C^3 continuity. Both schemes are presented in full.

We have investigated these schemes using the generating function formalism, which lends itself well to deriving sufficient conditions for subdivision schemes to be C^k. For binary schemes the subdivision step can be compactly written in a single equation

$$p_j^{i+1} = \sum_{k \in \mathbb{Z}} \alpha_{(2k-j)} p_k^i, \tag{1}$$

and similarly for ternary schemes

$$p_j^{i+1} = \sum_{k \in \mathbb{Z}} \alpha_{(3k-j)} p_k^i, \tag{2}$$

where $\alpha = (\alpha_j)$ is the mask of the scheme and p^i are the set of points after the i^{th} subdivision step. The principal results for the binary schemes are shown in Table 1 and can be compared with those for ternary schemes, shown in Table 2. Further analysis of the new schemes presented here, including the derivation of their exactness class, approximation order, and Holder exponent is given in [5].

The results for the 2-point interpolating schemes are trivial. [6] shows the derivation of the ternary 4-point interpolating scheme. [4] derives the results for the binary 4-point interpolating scheme and [8] derives the results for the binary 6-point interpolating scheme. Chaikin first proposed the binary 2-point approximating scheme in [1], which was shown to produce the quadratic b-spline at the limit [7] and it can be shown that the ternary 3-point approximating scheme produces the cubic B-spline at

Interpolating		
Scheme	Continuity	Mask
2-point	C^0	$\frac{1}{3}[1,2,3,2,1]$
3-point*	C^1	$[a,0,b,1-a-b,1,1-a-b,b,0,a]$
4-point	C^2	$[a_3,a_0,0,a_2,a_1,1,a_1,a_2,0,a_0,a_3]$
Approximating		
3-point*	C^2	$\frac{1}{27}[1,4,10,16,19,16,10,4,1]$

Tab. 2. Table showing results for ternary schemes where $a = b - \frac{3}{9}$, $a_0 = -\frac{1}{18} - \frac{1}{6}\mu$, $a_1 = \frac{13}{18} + \frac{1}{2}\mu$, $a_2 = \frac{7}{18} - \frac{1}{2}\mu$, $a_3 = -\frac{1}{18} + \frac{1}{6}\mu$, and $\frac{2}{9} < b < \frac{3}{9}$, $\frac{1}{9} < \mu < \frac{1}{15}$. * Indicates schemes presented and analyzed in this paper.

the limit and that the binary 3-point approximating scheme produces the quartic B-spline.

In the following we will describe the generating function formalism and how it is used to derive continuity.

§2. Generating Function Formalism

2.1 Binary Schemes

From the method of Dyn [3], we see that the subdivision step for binary schemes can be expressed in the generating function formalism as a simple multiplication of the corresponding symbols:

$$P^{i+1}(z) = \alpha(z)P^i(z^2), \tag{3}$$

where

$$P^i(z) = \sum_j p_j^i z^j, \alpha(z) = \sum_j \alpha_j z^j.$$

Sufficient conditions for C^k

Now we will state sufficient conditions for a binary scheme to be C^k. The proof is given in [3].

For any given binary subdivision scheme, S, with a mask α satisfying (4), we can prove $S^\infty P^0 \in C^k$ by first deriving the mask of $\frac{1}{2}S_{k+1}$ and then computing $||(\frac{1}{2}S_{k+1})^i||_\infty$ for $i = 1,2,3,\dots L$, where L is the first integer for which $||(\frac{1}{2}S_{k+1})^L||_\infty < 1$. If such an L exists and the mask of S_l satisfies (4) $\forall l \le k$ then $S^\infty P^0 \in C^k$.

$$\sum_{j\in\mathbb{Z}} \alpha_{2j} = 1, \sum_{j\in\mathbb{Z}} \alpha_{2j+1} = 1. \tag{4}$$

$$\left\|\frac{1}{2}S_{k+1}\right\|_{\infty} = \frac{1}{2}\max\left(\sum_{j\in\mathbb{Z}}\left|\alpha_{2j}^{(k+1)}\right|, \sum_{j\in\mathbb{Z}}\left|\alpha_{2j+1}^{(k+1)}\right|\right)$$

where

$$2z[\alpha^{(k)}(z)] = [\alpha^{(k+1)}(z)](1+z)$$
$$\Rightarrow \qquad 2\alpha_i^{(k)} = \alpha_{i-1}^{(k+1)} + \alpha_i^{(k+1)}$$

2.2 Ternary Schemes

Again following the method of Dyn [3], after some computation, we see that the subdivision step for ternary schemes can be expressed in the generating function formalism as a simple multiplication of the corresponding symbols:

$$P^{i+1}(z) = \alpha(z)P^i(z^3), \tag{5}$$

where

$$P^i(z) = \sum_j p_j^i z^j, \alpha(z) = \sum_j \alpha_j z^j.$$

Sufficient Conditions for C^k

Now we will state sufficient conditions for a ternary scheme to be C^k. The proof is given in [6].

For any given ternary subdivision scheme, S, with a mask α satisfying (6), we can prove $S^{\infty}P^0 \in C^k$ by first deriving the mask of $\frac{1}{3}S_{k+1}$ and then computing $||(\frac{1}{3}S_{k+1})^i||_{\infty}$ for $i = 1, 2, 3, \ldots L$, where L is the first integer for which $||(\frac{1}{3}S_{k+1})^L||_{\infty} < 1$. If such an L exists and the mask of S_l satisfies (6) $\forall l \leq k$ then $S^{\infty}P^0 \in C^k$.

$$\sum_{j\in\mathbb{Z}} \alpha_{3j} = 1, \sum_{j\in\mathbb{Z}} \alpha_{3j+1} = 1, \sum_{j\in\mathbb{Z}} \alpha_{3j+2} = 1. \tag{6}$$

$$\left\|\frac{1}{3}S_{k+1}\right\|_{\infty} = \frac{1}{3}\max\left(\sum_{j\in\mathbb{Z}}\left|\alpha_{3j}^{(k+1)}\right|, \sum_{j\in\mathbb{Z}}\left|\alpha_{3j+1}^{(k+1)}\right|, \sum_{j\in\mathbb{Z}}\left|\alpha_{3j+2}^{(k+1)}\right|\right)$$

where

$$3z^2[\alpha^{(k)}(z)] = [\alpha^{(k+1)}(z)](1+z+z^2)$$
$$\Rightarrow \qquad 3\alpha_i^{(k)} = \alpha_{i-2}^{(k+1)} + \alpha_{i-1}^{(k+1)} + \alpha_i^{(k+1)}$$

§3. Approximating 3-point Binary Subdivision

3.1 Continuity

This scheme can be easily derived from the quartic B-spline. Here we take a different approach. We start from the general form of a binary 3-point subdivision scheme. We then apply continuity requirements in order, showing that the quartic B-spline scheme is the only scheme of this type which has C^3-continuity, but that there are an infinite range of schemes with lower continuity.

There is no point in having a 3-point interpolating binary scheme, as such a scheme would reduce to the 2-point scheme: $\frac{1}{2}[1,2,1]$. However, a 3-point approximating binary scheme may be possible. This would be a corner-cutting scheme similar to the 2-point scheme: $\frac{1}{4}[1,3,3,1]$. Its mask is

$$\alpha = [a, b, 1-a-b, 1-a-b, b, a]$$

For C^0 continuity we require that the mask satisfy (4), which it does, and $\left|\left|\frac{1}{2}S_1\right|\right|_\infty < 1$.

$$\alpha^{(1)} = 2[\ldots,0,0,a,b-a,1-2b,b-a,a,0,0,\ldots] \qquad (7)$$

$$\Rightarrow \qquad \left|\left|\frac{1}{2}S_1\right|\right|_\infty = \max\left(|1-2b| + 2|a|, 2|b-a|\right) < 1 \qquad (8)$$

For C^1 continuity we require that $\alpha^{(1)}$ satisfy (4), which implies that $b = a + \frac{1}{4}$, and also $\left|\left|\frac{1}{2}S_2\right|\right|_\infty < 1$.

$$\alpha^{(2)} = 4[\ldots,0,0,a,\frac{1}{4}-a,\frac{1}{4}-a,a,0,0,\ldots] \qquad (9)$$

$$\Rightarrow \qquad \left|\left|\frac{1}{2}S_2\right|\right|_\infty = 2|a| + 2\left|\frac{1}{4}-a\right| < 1 \qquad (10)$$

For C^2 continuity we require that $\alpha^{(2)}$ satisfy (4), which is true, and also $\left|\left|\frac{1}{2}S_3\right|\right|_\infty < 1$.

$$\alpha^{(3)} = 8[\ldots,0,0,a,\frac{1}{4}-2a,a,0,0,\ldots]$$

$$\Rightarrow \qquad \left|\left|\frac{1}{2}S_3\right|\right|_\infty = \max\left(|8a|, |1-8a|\right) < 1$$

which implies that $0 < a < \frac{1}{8}$.

For C^3 continuity we require that $\alpha^{(3)}$ satisfy (4), which implies that $a = \frac{1}{16}$, which incidentally meets the criterion in equations (8) and (10), and also $\left|\left|\frac{1}{2}S_4\right|\right|_\infty < 1$.

$$\alpha^{(4)} = [\ldots, 0, 0, 1, 1, 0, 0, \ldots]$$
$$\Rightarrow \qquad \left|\left|\frac{1}{2}S_4\right|\right|_\infty = \max\left(\frac{1}{2}, \frac{1}{2}\right) < 1$$

To go to C^4 continuity we require that $\alpha^{(4)}$ satisfy (4), which it does, and also $\left|\left|\frac{1}{2}S_5\right|\right|_\infty < 1$, which it does not:

$$\alpha^{(5)} = 8[\ldots, 0, 0, 2, 0, 0, \ldots]$$
$$\Rightarrow \qquad \left|\left|\frac{1}{2}S_5\right|\right|_\infty = 1$$

Thus the limit curve for the binary scheme with the mask $\alpha = \frac{1}{16}[1, 5, 10, 10, 5, 1]$ has C^3 continuity.

§4. Interpolating 3-point Ternary Subdivision

4.1 Continuity

For this scheme we have

$$\alpha = [\ldots, 0, 0, a, 0, b, 1-a-b, 1, 1-a-b, b, 0, a, 0, 0, \ldots] \qquad (11)$$
$$\alpha^{(1)} = 3[\ldots, 0, 0, a, -a, b, 1-2b, b, -a, a, 0, 0, \ldots] \qquad (12)$$

It is easy to verify that α satisfies (6).

If

$$\left|\left|\frac{1}{3}S_1\right|\right|_\infty = \max\left(|1-2b| + 2|a|, |a| + |b|, |a| + |b|\right) < 1 \qquad (13)$$

then this scheme has C^0 continuity.

Now for C^1 continuity we first need $\alpha^{(1)}$ to satisfy (6). This implies

$$a = b - \frac{1}{3}. \qquad (14)$$

and so we have

$$\alpha^{(1)} = 3[\ldots, 0, 0, b - \frac{1}{3}, \frac{1}{3} - b, b, 1-2b, b, \frac{1}{3} - b, b - \frac{1}{3}, 0, 0, \ldots]$$
$$\alpha^{(2)} = 9[\ldots, 0, 0, b - \frac{1}{3}, \frac{2}{3} - 2b, 2b - \frac{1}{3}, \frac{2}{3} - 2b, b - \frac{1}{3}, 0, 0, \ldots]$$

If

$$\left|\left|\frac{1}{3}S_2\right|\right|_\infty = \max\left(9\left|b - \frac{1}{3}\right|, 9\left|b - \frac{1}{3}\right|, 3\left|2b - \frac{1}{3}\right|\right) < 1 \qquad (15)$$

then we have C^1 continuity.

$$\frac{2}{9} < b < \frac{3}{9}, a = b - \frac{3}{9}$$

satisfies (13) and (15).

For C^2 continuity we would require $\alpha^{(2)}$ to satisfy (6). This implies $b = \frac{2}{9}$, but with this value $||\frac{1}{3}S_2||_\infty = 1$. For $b = \frac{2}{9}$

$$\alpha^{(2)} = [\ldots, 0, 0, -1, 2, 1, 2, -1, 0, 0, \ldots]$$
$$(\alpha^{(2)})^2 = [\ldots, 0, 0, 1, -4, 2, 0, 11, 0, 2, -4, 1, 0, 0, \ldots]$$

* this vector is incorrect in the printed version, see the footnote for the corrected version

Hence $\left|\left|\left(\frac{1}{3}S_2\right)^2\right|\right|_\infty = \frac{19}{9}$. Furthermore it can be shown that

* likewise, this value is 1, not 19/9

$$\left|\left|\left(\frac{1}{3}S_2\right)^n\right|\right|_\infty > 1 \ \forall n \in \mathbb{N}.$$

* and the greater than sign should be greater than or equal to

Thus there is no C^2 3-point interpolating ternary subdivision scheme.

Hence a C^1 ternary 3-point interpolating subdivision scheme can be defined by (11), where $\frac{2}{9} < b < \frac{3}{9}$ and $a = b - \frac{1}{3}$.

§5. Approximating 3-point Ternary Subdivision

5.1 Continuity

There are several ways to arrive at this scheme. One is through the generating function formalism as followed above, another method is to use the matrix formalism, and finally it can be arrived at from the cubic B-spline itself. Here we will just prove the continuity of the scheme using the generating function formalism.

For this scheme we have

$$\alpha = \frac{1}{27}[\ldots, 0, 0, 1, 4, 10, 16, 19, 16, 10, 4, 1, 0, 0, \ldots]$$
$$\alpha^{(1)} = \frac{1}{9}[\ldots, 0, 0, 1, 3, 6, 7, 6, 3, 1, 0, 0, \ldots]$$

It is easy to verify that α satisfies (6).

$$\left|\left|\frac{1}{3}S_1\right|\right|_\infty = \max\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) < 1$$

Hence this scheme has C^0 continuity.

* the correct vector is: [...,0,0,1,-2,-1,-4,5,2,3,0,1,0,3,2,5,-4,-1,-2,1,0,0,...]

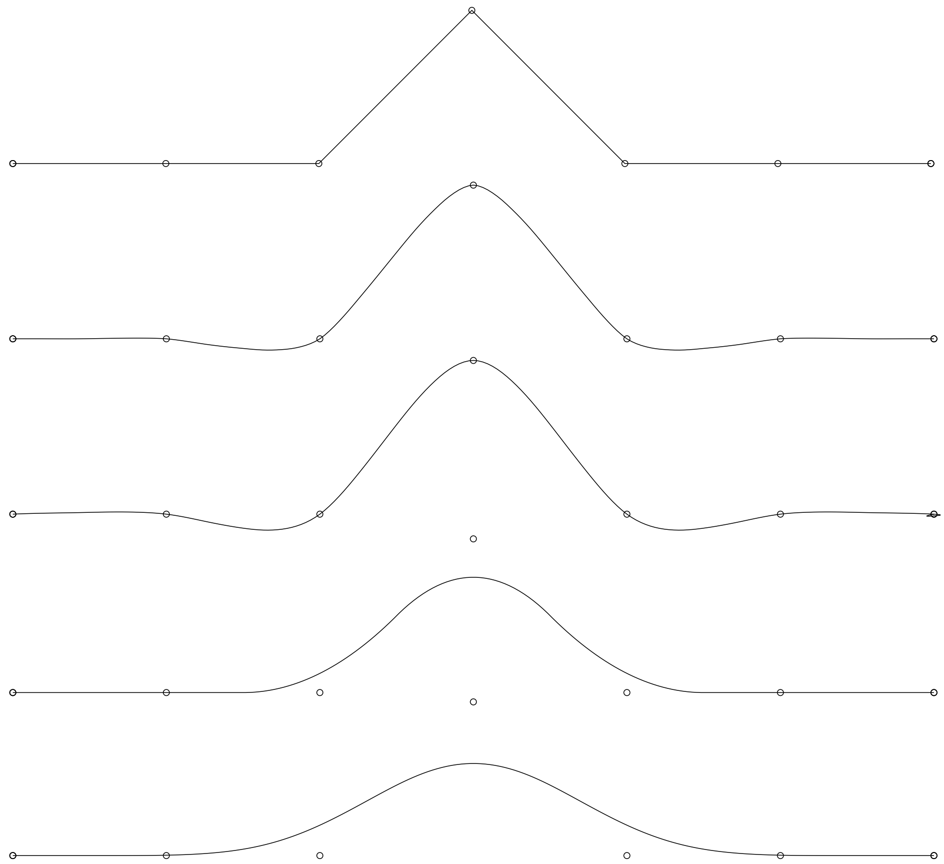

Fig. 1. Fundamental functions for the binary schemes. Top to bottom: 2-point interpolating, 4-point interpolating, 6-point interpolating ($\theta = 0.01$), 2-point approximating, 3-point approximating.

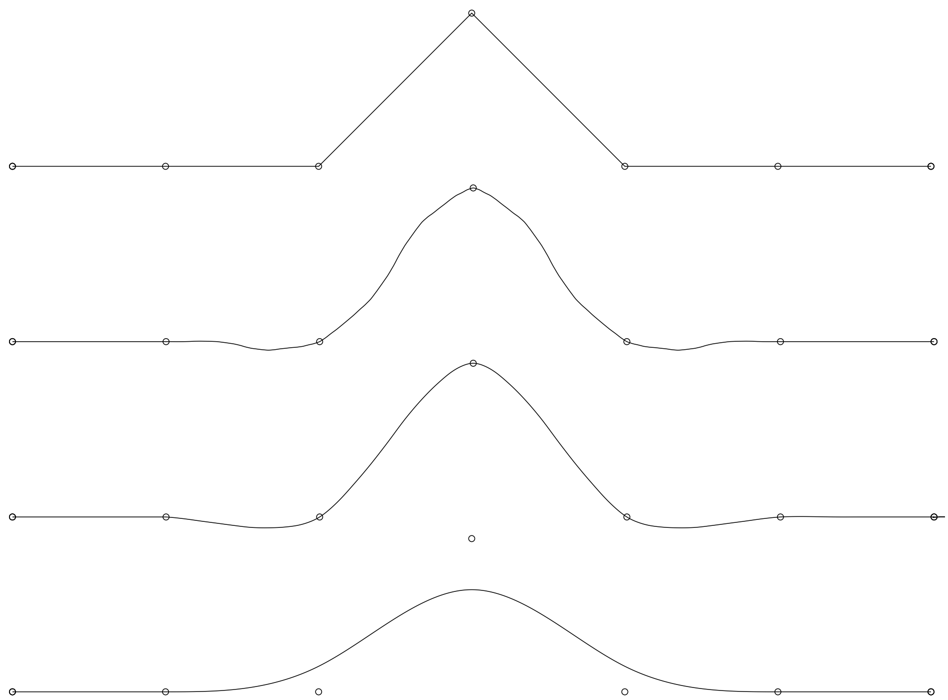

Fig. 2. Fundamental functions for the ternary schemes. Top to bottom: 2-point interpolating, 3-point interpolating ($b = 5/18$), 4-point interpolating ($\mu = 4/45$), 3-point approximating.

Now for C^1 continuity we first need $\alpha^{(1)}$ to satisfy (6), which it does. Now

$$\alpha^{(2)} = \frac{1}{3}[\ldots,0,0,1,2,3,2,1,0,0,\ldots]$$
$$\Rightarrow \qquad \left|\left|\frac{1}{3}S_2\right|\right|_\infty = \max\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right) < 1$$

Hence this scheme has C^1 continuity.

Now for C^2 continuity we first need $\alpha^{(2)}$ to satisfy (6), which it does. Now

$$\alpha^{(3)} = [\ldots,0,0,1,1,1,0,0,\ldots]$$
$$\Rightarrow \qquad \left|\left|\frac{1}{3}S_3\right|\right|_\infty = \max\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right) < 1$$

Hence this scheme has C^2 continuity.

Now for C^3 continuity we first need $\alpha^{(3)}$ to satisfy (6), which it does. But

$$\alpha^{(4)} = 3[\ldots,0,0,1,0,0,\ldots]$$
$$\Rightarrow \qquad \left|\left|\frac{1}{3}S_3\right|\right|_\infty = \max(1,0,0) \geq 1$$
$$(\alpha^{(4)})^2 = 9[\ldots,0,0,1,0,0,\ldots]$$
$$\Rightarrow \qquad \left|\left|\left(\frac{1}{3}S_3\right)^2\right|\right|_\infty = \max(1,0,0) \geq 1$$

Hence this scheme does not have C^3 continuity.

§6. Examples and Conclusion

We have used the generating function formalism to analyze the continuity properties of univariate ternary and binary subdivision schemes. Figures 1 and 2 show the fundamental functions for all the schemes shown in Tables 1 and 2. We can see from these figures that, for interpolating schemes, the ternary schemes have a smaller support than their binary counterparts.

These results show that we can achieve higher smoothness and smaller support for interpolatory schemes by going from binary to ternary. More work needs to be done to see if this trend continues for higher arities.

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