Ternary and Three Point Univariate Subdivision Schemes

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Introduction

Most work in the area of subdivision schemes has considered binary schemes with an even number of control points. Following a similar argument to that used in [2], we decided to investigate schemes with an odd number of control points, specifically 3-point schemes. This led to a more general investigation of ternary subdivision schemes. Ternary univariate schemes are important for Kobbelt's $\sqrt{3}$-scheme [7] at the boundary (see Figure 7).

For symmetry reasons, it is obvious that an interpolating binary subdivision scheme which utilizes the closest k points, for k odd, reduces to a scheme which utilizes just the closest $k-1$ points, $k-1$ even. There is thus no 3-point interpolating binary subdivision scheme. Ternary subdivision, on the other hand, does allow for an interpolating 3-point subdivision scheme. A family of such schemes is shown to exist and have C^1 continuity. Further investigation led to discovery of a family of interpolating 4-point ternary subdivision schemes which have C^2 continuity [6].

Investigation of approximating 3-point schemes has led to two interesting subdivision schemes. An approximating 3-point ternary scheme has been found and can be shown to have C^2 continuity. An approximating 3-point binary scheme, which uses corner-cutting similar in spirit to the 2-point scheme $\frac{1}{4}[1,3,3,1]$, was derived and can shown to have C^3 continuity.

Ternary Interpolating		
Scheme	Highest continuity	Mask(α)
2-point	C^0	$\frac{1}{3}[1,2,3,2,1]$
3-point	C^1	$[a,0,b,1-a-b,1,1-a-b,b,0,a]$
4-point	C^2	$[a_3,a_0,0,a_2,a_1,1,a_1,a_2,0,a_0,a_3]$
Ternary Approximating		
3-point	C^2	$\frac{1}{27}[1,4,10,16,19,16,10,4,1]$

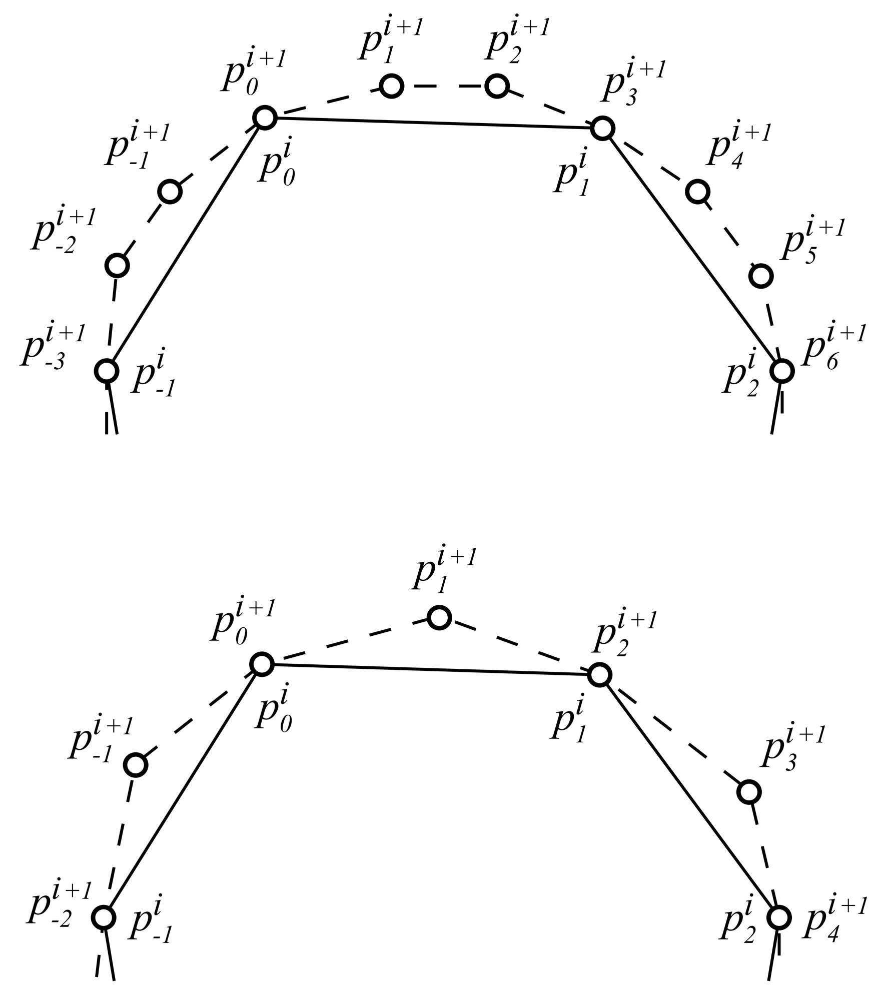
where

$$a = b - \frac{3}{9}$$
$$a_0 = -\frac{1}{18} - \frac{1}{6}\mu$$
$$a_1 = \frac{13}{18} + \frac{1}{2}\mu$$
$$a_2 = \frac{7}{18} - \frac{1}{2}\mu$$
$$a_3 = -\frac{1}{18} + \frac{1}{6}\mu$$

and

$$\frac{2}{9} < b < \frac{3}{9}$$
$$\frac{1}{9} < \mu < \frac{1}{15}$$

Figure 2: Table showing results for ternary schemes. The new schemes are in red.



$$p_j^{i+1} = \sum_{k\in\mathbb{Z}} \alpha_{(3k-j)} p_k^i$$

$$p_j^{i+1} = \sum_{k\in\mathbb{Z}} \alpha_{(2k-j)} p_k^i$$

Figure 1: Ternary (top) and binary (bottom) subdivision. A polygon $P^i = (p_j^i)$ (solid lines) is mapped to a refined polygon $P^{i+1} = (p_j^{i+1})$ (dashed lines). These examples show interpolatory schemes: $\alpha_0 = 1$.

We have investigated these schemes using the generating function formalism, which lends itself well to deriving sufficient conditions for subdivision schemes to be C^k. For binary schemes the subdivision step can be compactly written in a single equation

$$p_j^{i+1} = \sum_{k\in\mathbb{Z}} \alpha_{(2k-j)} p_k^i$$

and similarly for ternary schemes

$$p_j^{i+1} = \sum_{k\in\mathbb{Z}} \alpha_{(3k-j)} p_k^i$$

where $\alpha = (\alpha_j)$ is the mask of the scheme and p^i are the set of points after the i^{th} subdivision step (see Figure 1). The principal results for the ternary schemes are tabulated in Figure 2 and compared with those for binary schemes tabulated in Figure 3.

The results for the 2-point interpolating schemes are trivial. [6] shows the derivation of the ternary 4-point interpolating scheme. [4] derives the results for the binary 4-point interpolating scheme and [9] derives the results for the binary 6-point interpolating scheme. Chaikin first proposed the binary 2-point approximating scheme in [1], which was shown to produce the quadratic B-spline at the limit [8] and it can be shown that the ternary 3-point approximating scheme produces the cubic B-spline at the limit and that the binary 3-point approximating scheme produces the quartic B-spline.

Binary Interpolating		
Scheme	Highest continuity	Mask(α)
2-point	C^0	$\frac{1}{2}[1,2,1]$
4-point	C^1	$\frac{1}{16}[-1,0,9,16,9,0,-1]$
6-point	C^2	$[\theta,0,-3\theta-\frac{1}{16},0,2\theta+\frac{9}{16},1,2\theta+\frac{9}{16},0,-3\theta-\frac{1}{16},0,\theta]$
Binary Approximating		
2-point	C^1	$\frac{1}{4}[1,3,3,1]$
3-point	C^3	$\frac{1}{16}[1,5,10,10,5,1]$

where $0 < \theta < 0.02$.

Figure 3: Table showing results for binary schemes. The new scheme is in red.

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Ternary and Three Point Univariate Subdivision Schemes...continued

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Sufficient conditions for binary schemes

From the method of Dyn [3], after some computation we, see that the subdivision step for binary schemes can be expressed in the generating function formalism as a simple multiplication of the corresponding symbols:

$$P^{i+1}(z) = \alpha(z)P^i(z^2), \tag{1}$$

where

$$P^i(z) = \sum_j p^i_j z^j, \alpha(z) = \sum_j \alpha_j z^j. \tag{2}$$

For any given binary subdivision scheme, S, with a mask α satisfying (3), we can prove $S^\infty P^0 \in C^k$ by first deriving the mask of $\frac{1}{2}S_{k+1}$ and then computing $||(\frac{1}{2}S_{k+1})^i||_\infty$ for $i = 1, 2, 3, \ldots L$, where L is the first integer for which $||(\frac{1}{2}S_{k+1})^L||_\infty < 1$. If such an L exists and the mask of S_l satisfies (3) $\forall l \leq k$ then $S^\infty P^0 \in C^k$. The proof is given in [3].

$$\sum_{j\in\mathbb{Z}} \alpha_{2j} = 1, \sum_{j\in\mathbb{Z}} \alpha_{2j+1} = 1. \tag{3}$$

$$\left\|\frac{1}{2}S_{k+1}\right\|_\infty = \frac{1}{2}\max\left(\sum_{j\in\mathbb{Z}}\left|\alpha^{(k+1)}_{2j}\right|, \sum_{j\in\mathbb{Z}}\left|\alpha^{(k+1)}_{2j+1}\right|\right) \tag{4}$$

where

$$2z[\alpha^{(k)}(z)] = [\alpha^{(k+1)}(z)](1+z) \tag{5}$$

$$\Rightarrow \quad 2\alpha^{(k)}_i = \alpha^{(k+1)}_{i-1} + \alpha^{(k+1)}_i \tag{6}$$

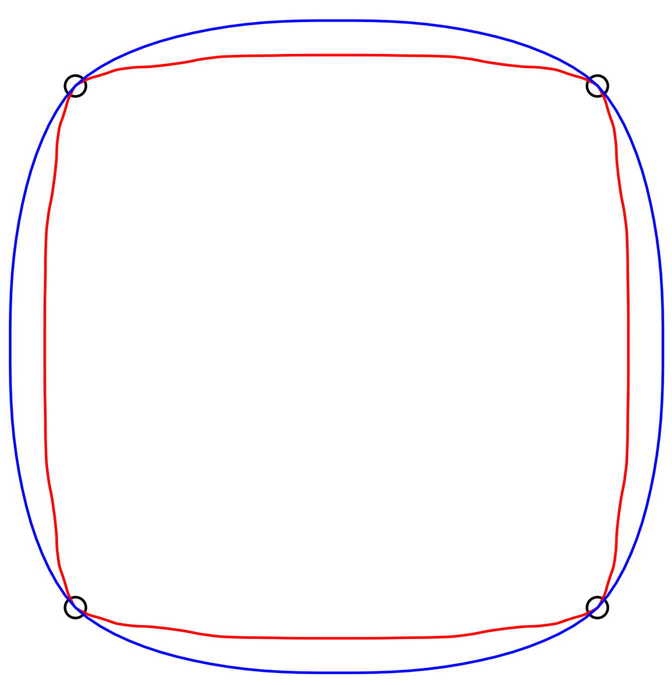

Figure 4: C^1 interpolating schemes. Ternary 3-point with $b = 5/18$ (red) and binary 4-point (blue).

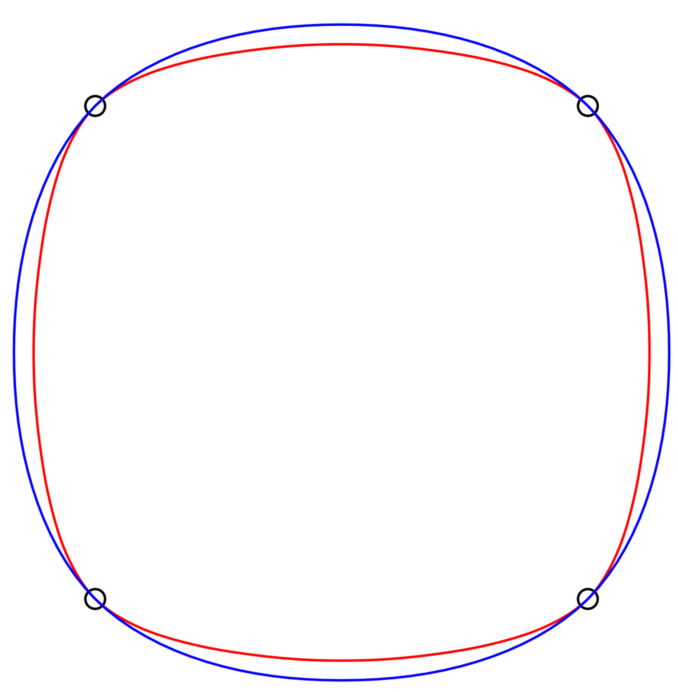

Figure 5: C^2 interpolating schemes. Ternary 4-point with $\mu = 4/45$ (red) and binary 6-point with $\theta = 0.01$ (blue).

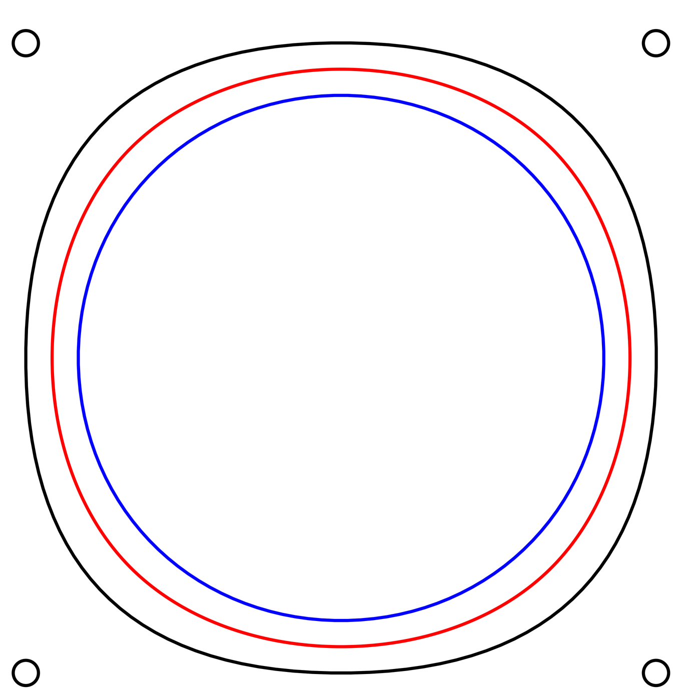

Figure 6: Approximating schemes. Binary 2-point (black), ternary 3-point (red), binary 3-point (blue).

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Sufficient conditions for ternary schemes

The subdivision step for ternary schemes can be expressed in the generating function formalism as a multiplication of the corresponding symbols:

$$P^{i+1}(z) = \alpha(z)P^i(z^3), \tag{7}$$

where

$$P^i(z) = \sum_j p^i_j z^j, \alpha(z) = \sum_j \alpha_j z^j. \tag{8}$$

For any given ternary subdivision scheme, S, with a mask α satisfying (9), we can prove $S^\infty P^0 \in C^k$ by first deriving the mask of $\frac{1}{3}S_{k+1}$ and then computing $||(\frac{1}{3}S_{k+1})^i||_\infty$ for $i = 1, 2, 3, \ldots L$, where L is the first integer for which $||(\frac{1}{3}S_{k+1})^L||_\infty < 1$. If such an L exists and the mask of S_l satisfies (9) $\forall l \leq k$ then $S^\infty P^0 \in C^k$. The proof is given in [6].

$$\sum_{j\in\mathbb{Z}} \alpha_{3j} = 1, \sum_{j\in\mathbb{Z}} \alpha_{3j+1} = 1, \sum_{j\in\mathbb{Z}} \alpha_{3j+2} = 1. \tag{9}$$

$$\left\|\frac{1}{3}S_{k+1}\right\|_\infty = \frac{1}{3}\max\left(\sum_{j\in\mathbb{Z}}\left|\alpha^{(k+1)}_{3j}\right|, \sum_{j\in\mathbb{Z}}\left|\alpha^{(k+1)}_{3j+1}\right|, \sum_{j\in\mathbb{Z}}\left|\alpha^{(k+1)}_{3j+2}\right|\right) \tag{10}$$

where

$$3z^2[\alpha^{(k)}(z)] = [\alpha^{(k+1)}(z)](1+z+z^2) \tag{11}$$

$$\Rightarrow \quad 3\alpha^{(k)}_i = \alpha^{(k+1)}_{i-2} + \alpha^{(k+1)}_{i-1} + \alpha^{(k+1)}_i \tag{12}$$

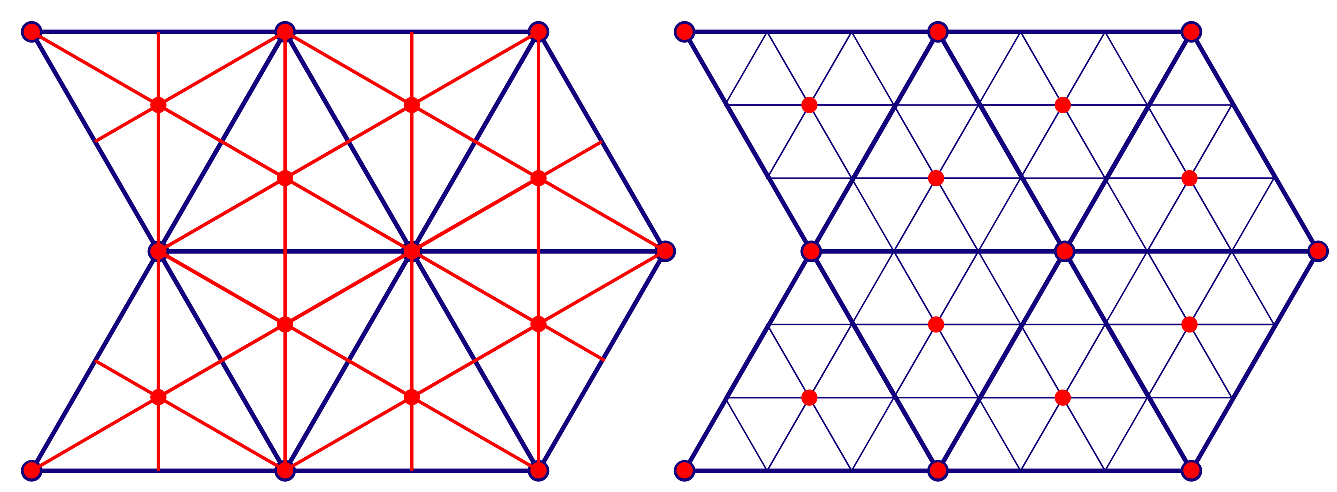

Figure 7: $\sqrt{3}$-scheme after 1 refinement (left) and two refinements (right). The boundary after two refinements is ternary.

Conclusions

More details, including the proofs for the smoothness of our results can be seen in [5]. These results show that we achieve higher smoothness and smaller support by going from binary to ternary schemes. More work needs to be done to find out whether this trend continues for higher arities.

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