An Heuristic Analysis of the Classification of Bivariate Subdivision Schemes

Neil A. Dodgson

University of Cambridge Computer Laboratory
15 J. J. Thomson Avenue, Cambridge, UK CB3 0FD
nad@cl.cam.ac.uk

Abstract. Alexa [1] and Ivrissimtzis *et al.* [2] have proposed a classification mechanism for bivariate subdivision schemes. Alexa considers triangular primal schemes, Ivrissimtzis *et al.* generalise this both to quadrilateral and hexagonal meshes and to dual and mixed schemes. I summarise this classification and then proceed to analyse it in order to determine which classes of subdivision scheme are likely to contain useful members. My aim is to ascertain whether there are any potentially useful classes which have not yet been investigated or whether we can say, with reasonable confidence, that all of the useful classes have already been considered.

I apply heuristics related to the mappings of element types (vertices, face centres, and mid-edges) to one another, to the preservation of symmetries, to the alignment of meshes at different subdivision levels, and to the size of the overall subdivision mask. My conclusion is that there are only a small number of useful classes and that most of these have already been investigated in terms of linear, stationary subdivision schemes. There is some space for further work, particularly in the investigation of whether there are useful ternary linear, stationary subdivision schemes, but it appears that future advances are more likely to lie elsewhere.

1 Introduction

Alexa [1] and Ivrissimtzis *et al.* [2] propose a classification of subdivision schemes. Alexa classifies all triangular primal schemes. Ivrissimtzis *et al.* extend this both to quadrilateral and hexagonal base meshes and to dual and mixed schemes (this terminology is explained later in this section). The extension to quadrilateral meshes is based on Sloan's work on 2D lattices [3].

While this classification tells of the existence of many classes of subdivision scheme, it does not give any indication as to which classes are likely to contain useful schemes. This paper analyses Ivrissimtzis *et al.*'s classification with the intention of determining which classes are likely to contain useful (stationary, linear) subdivision schemes and which classes are unlikely to contain useful schemes. I expect that there will be an indeterminate region between those classes which clearly contain useful schemes and those classes which clearly do not. I assume that the reader is familiar with subdivision [4].

Mathematics of Surfaces XI, R. Martin, H. Bez, M. Sabin (editors), LNCS 3604, Springer-Verlag, 2005, pp. 161 -183, ISBN 3-540-28225-4

Subdivision schemes may be classified in a variety of ways. Ivrissimtzis, Sabin and I use a hierarchy of detail, where the top level classes encompass many subdivision schemes, while the lowest level precisely specifies a single scheme. The hierarchy has the following levels (this is an expanded form of the list given by Ivrissimtzis *et al.* [2]).

Base mesh type. This is the base mesh in the regular case. Most subdivision schemes are based on either a quadrilateral or a triangular mesh. It is also possible to base a scheme on an hexagonal mesh, this being the only other regular monohedral tiling of the plane [5], or on one of the semi-regular tilings of the plane.

Mapping. This concerns how vertices, face centres, and mid-edges map to one another from one level of subdivision to the next. Face centres and mid-edges refer to these points in a regular tiling of the plane. If one applies subdivision to a regular tiling of the plane, the mapping is exact. In the case of a general mesh in 3D space, we can think of the regular tiling of the plane as a parameterization of the actual mesh. Ivrissimtzis *et al.* [2] classify schemes based on whether vertices map to vertices or to face centres. In this paper I extend this to consider what elements are mapped to by face centres and to consider also the mappings of mid-edges.

Arity. This describes how the source grid maps to the subdivided grid in the regular case. It can be represented either as a scalar, representing the ratio of the lengths of edges in the source and subdivided grids, or as an ordered pair, (n, m), giving the relative position, in the coordinate system of the subdivided grid, of one source vertex with respect to an adjacent source vertex (see Fig. 1); in the case of the hexagonal grid, of the position of one source face centre with respect to an adjacent one (see Fig. 2). Without loss of generality we can take $n > 0$ and $0 \leq m \leq n$. Thus $(2, 0)$ represents binary subdivision (e.g. Catmull-Clark [6], Doo-Sabin [7], Loop [8]), while $(1, 1)$ represents the $\sqrt{2}$ class for quadrilateral grids (e.g. simplest [9], Peters-Shiue [10], Velho [11, 12]) and the $\sqrt{3}$ class for triangular and hexagonal grids (e.g. Kobbelt's $\sqrt{3}$ [13], hexagon-by-three [14]). Examples are shown in Fig. 3.

Footprint. Having chosen values for the above three, the next level is to specify which new vertices are affected by a given source vertex in the regular case. This corresponds to specifying which coefficients in the subdivision mask are non-zero. A larger footprint gives greater freedom in choice of coefficients but also greater computation and increased difficulties in handling extraordinary points. Of the well-known published schemes, simplest [9] has the smallest footprint (4 vertices) while Catmull-Clark [6], butterfly [15], and Kobbelt [16] have the largest (25 vertices in each case). Amongst more recent schemes, ternary Loop [17] has 61 non-zero coefficients and interpolating ternary triangular [18] has up to 85. I note that the terminology is not consistent in the literature, so it is worth saying that I am using Sabin's definition of the term *mask* [19] where the mask shows the contributions made to each new vertex

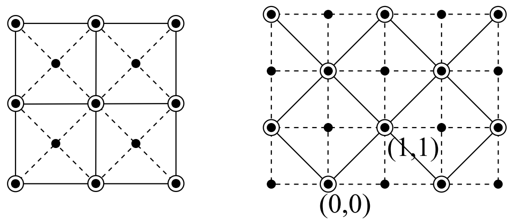

Fig. 1. Open circles are source vertices; black dots are subdivided vertices. The solid lines are the source mesh; the dashed lines are the subdivided mesh. At left is a visualisation of $QP(1,1)$ subdivision as we usually think of it: a new vertex is introduced at the centre of each quadrilateral, the old vertices are adjusted, and the new grid is constructed as shown. At right is an equivalent visualisation, this time with the subdivided grid aligned horizontally and vertically. If the edges of the subdivided mesh are assigned unit length then this is the coordinate system used by Ivrissimtzis *et al.* [2] for quadrilateral meshes, which is used throughout this paper.

by a given old vertex, c.f. the *stencils* where a stencil shows the contributions made by each old vertex to a given new vertex.

Mask coefficients. The next step is to decide what values the coefficients should have. For B-spline based and box-spline based schemes, there is no freedom beyond choosing the particular spline basis, as the coefficients must be derived from the spline on which they are based. Other schemes have more freedom (e.g. butterfly [15], Kobbelt [16], interpolating ternary triangular [18]). Amongst other things, the choice of coefficients determines whether the scheme is interpolating or approximating. Interpolating schemes (e.g. butterfly [15]) are those where the limit surface is constrained to pass through the source vertices. Approximating schemes (e.g. Loop [8]) do not have this constraint.

Extraordinary cases, boundaries, and creases. The final step is to handle the extraordinary cases. This is the step which requires a significant amount of careful thought and analysis. Some schemes have more than one proposed method for handling extraordinary cases. For example, the schemes based on the bivariate quadratic and cubic B-splines are commonly known as Doo-Sabin and Catmull-Clark subdivision respectively but, in fact, each of them has two variant mechanisms for handling extraordinary cases: one proposed by Catmull and Clark [6] and one proposed by Doo and Sabin [7]. Boundaries of the mesh must also be handled as special cases as must creases [4] in the mesh which are internal edges along which a designer wants reduced continuity.

Alexa [1] and Ivrissimtzis *et al.* [2] consider the top three levels of this hierarchy. This paper analyses that classification in order to ascertain which classes are likely to contain useful schemes.

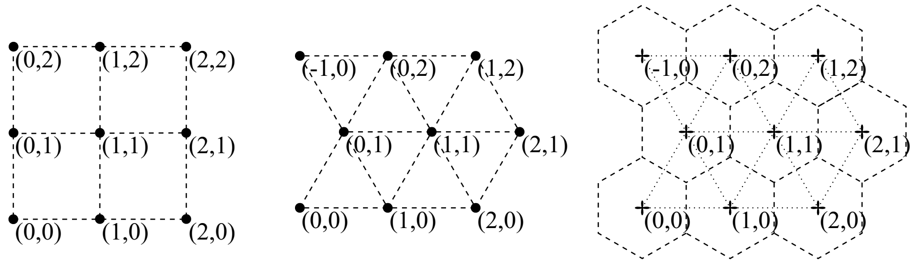

Fig. 2. The coordinate systems of the three mesh types. The quadrilateral mesh has the conventional coordinate system. Each edge is of unit length. The triangular mesh has axes at an angle $\pi/3$ to one another, with all edges of unit length. The hexagonal mesh is more complex. As with the triangular mesh, the axes are at an angle $\pi/3$ to one another, but it is the face centres which are at integer coordinates; edges are of length one-third, and vertices are at $\left(x + \frac{1}{3}, y + \frac{1}{3}\right)$, $\left(x + \frac{2}{3}, y + \frac{2}{3}\right)$, $x, y \in \mathbb{Z}$. This makes the hexagonal mesh a precise dual of the triangular mesh, as illustrated in the figure. See Appendix A for more on this coordinate system.

2 Summary of the Classification Notation

Ivrissimtzis *et al.* [2] use notation of the form $AB(n, m)$, where A is the base mesh type, B the mapping, and (n, m) the arity. Occasionally it is convenient to use $A(n, m)$ as a shorthand for all classes with the same base mesh type and arity. The coordinate systems are illustrated in Fig. 2 and example classes are shown in Fig. 3.

A can be Q (quadrilateral), T (triangular) or H (hexagonal). The right-triangle based schemes (e.g. Velho's 4-8 scheme [11, 12]) are regarded as Q schemes, because the vertices lie on the quadrilateral grid in the regular case. The right-triangle tiling, its dual (the octagon-square semi-regular tiling), and other semi-regular tilings, could be considered as primitive base mesh types in their own right, but Ivrissimtzis *et al.* [2] limit the classification to the three regular base tilings.

B can be P (primal), D (dual) or M (mixed) where *primal* means that all vertices map to vertices, *dual* that all vertices map to face centres, and *mixed* that vertices map to a combination of vertices and face centres. This classification as 'primal' and 'dual' arises from the $(2, 0)$ classes for which the terminology is well known [20] and where it is related to the concept of the dual graph. Sabin [19] points out that the classification as 'primal' and 'dual' is not necessarily particularly satisfactory for the general case. For example, in $Q(2, 0)$ classes it is related to the concept of face-splitting (primal) or vertex-splitting (dual), but this face- or vertex-splitting relationship fails for most other arities. In particular, Oswald and Schröder note that it fails for $(1, 1)$ classes [21]. The limitations of this classification are explored further in Sect. 3.3.

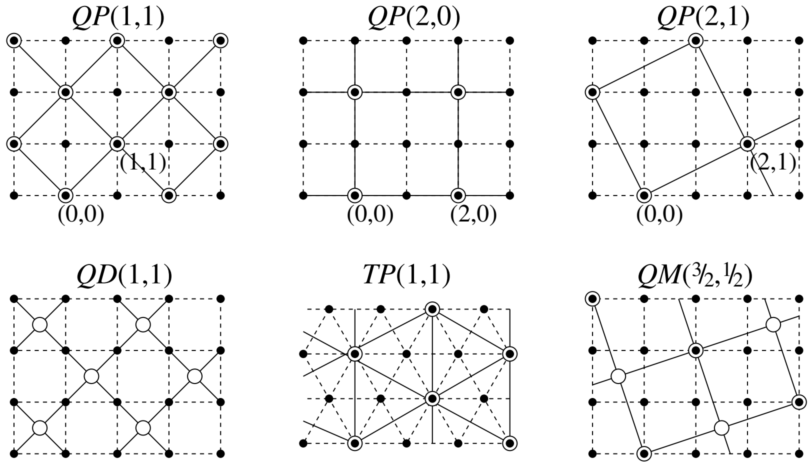

Fig. 3. Some example classes. Open circles are source vertices; black dots are subdivided vertices. The solid lines are the source mesh; the dashed lines are the subdivided mesh. Note how the (n, m) notation gives the coordinates, in the coordinate system of the subdivided grid, of an adjacent source vertex with respect to an arbitrary origin source vertex; to illustrate this, the top line of examples has an origin and the appropriate adjacent source vertex explicitly labelled with their coordinates.

(n, m) is the arity, as described in Sect. 1. There are certain quirks in the specification of arity for the TM and HM (triangular mixed and hexagonal mixed) classes, which I will gloss over here as they have no impact on the conclusions of this paper (for details see Ivrissimtzis *et al.* [2]). The term *arity* can refer to either (n, m) or to the length of the vector, which is $\sqrt{n^2 + m^2}$ for Q and $\sqrt{(n + m/2)^2 + (\sqrt{3}n/2)^2}$ for T and H.

The classes of arity $(1, 0)$ represent schemes which do not subdivide. These can be identity schemes, where the mesh does not change at all, or other point-processing schemes comparable with filters used in image processing. The simplest application of these would be mesh smoothing.

There is an interesting case with the lowest possible arity class considered by Ivrissimtzis *et al.* [2], which is the class of $QM\left(\frac{1}{2}, \frac{1}{2}\right)$ schemes. The arity (length of the (n, m) vector) is $\frac{1}{\sqrt{2}}$, which is less than unity, and therefore this class represents decimation schemes, rather than subdivision schemes.

3 Heuristic Analysis

This classification allows for a large number of potential subdivision schemes. This paper asks which of these classes are likely to contain useful schemes and thus reward further investigation and, conversely, which are likely to have unresolvable problems. To facilitate a partition into usable and unusable classes, I sequentially introduce heuristics, each providing more stringent requirements on what is meant by "usable".

An heuristic is a rule of thumb, a guideline which helps us to consider only the useful alternatives. In subdivision, one early heuristic appears to have been "only binary schemes are worth considering." This apparant heuristic has been seriously challenged by the discovery and development of $\sqrt{2}$ [9, 11, 12, 22, 23], $\sqrt{3}$ [13, 24] and ternary [17, 25, 18] schemes. They have not, however, completely invalidated it because all commercial systems are based on binary schemes. An up-to-date version of this example heuristic would therefore seem to be something like "only schemes based on Catmull-Clark or Loop are worth considering in a commercial context." As with all heuristics, it is possible to argue both for and against it.

This paper sets out a number of heuristics which are designed to reduce the enormous number of potential schemes which are allowed for by the classification mechanism. None of the heuristics is a hard and fast rule and not all of them have a solid mathematical justification. Nevertheless, I believe that they are rules of which all practitioners of subdivision become aware, whether consciously or not. It may well be that, as with the "only binary schemes are useful" heuristic, some of these heuristics will prove to be false guides. The commentary following each heuristic therefore incorporates discussion of those situations in which the heuristic appears to be a less than perfect guide.

3.1 Heuristics Implicit in Ivrissimtzis *et al.*'s Classification

The first two heuristics are implicit in Ivrissimtzis *et al.*'s [2] classification system. The classification is thus already making assumptions about which types of subdivision schemes are likely to prove useful. For comparison, Han has produced a much more restricted classification system for subdivision schemes [26] in which he implicitly assumes that Heuristics 1–6 are true.

Heuristic 1. *Only regular monohedral tilings of the plane are useful as base meshes.*

This limits the base mesh in the regular case to being quadrilateral, triangular, or hexagonal, with the individual polygons being regular. There are subdivision schemes which appear to be based on a right-triangle mesh [11, 12] but these can be treated as Q schemes, because the vertices lie on the quadrilateral grid in the regular case; the right-triangle concept simply serves to make the explanation and implementation of the scheme somewhat easier in practice. The right-triangle tiling, its dual (the octagon-square semi-regular tiling), and other semi-regular tilings, could be considered as primitive base mesh types in their own right. In addition to semi-regular tilings it may be possible to create a subdivision scheme based on an aperiodic tiling, such as a Penrose tiling [27]. In any semi-regular or aperiodic case there would seem to be some difficulty in specifying the base mesh for an object and in extending the subdivision scheme to handle extraordinary cases, boundaries, and creases. Nevertheless, Ivrissimtzis, Claes, and I undertook some preliminary work on octagon-square subdivision schemes in 2003. It was clear from this that some sort of octagon-square subdivision scheme is possible, although the above difficulties would have to be faced; in particular it is difficult to see how to handle extraordinary faces with an odd number of edges. It was also clear that the vertices do not lie on one of the three regular meshes, unlike the right-triangle mesh whose vertices lie on the quadrilateral mesh. There may be some advantage in investigating such schemes but they seem to pose immense difficulties. Furthermore, the classification mechanism does not admit such schemes. I do not consider them further.

Heuristic 2. *Every vertex at one level of subdivision must map to either a vertex or a face centre at the next level.*

Ivrissimtzis *et al.*'s [2] classification assumes this. The second letter in the classification indicates whether the mapping is to vertices (P), face centres (D) or a mixture (M). The initial motivation for this was from consideration of primal and dual binary schemes which have either a P or D behaviour. I conjecture that it would be possible to construct a subdivision scheme where vertices at one level map to some feature other than a vertex or face centre at the next level, but that it is likely that such a scheme would not prove useful because, as described under Heuristic 3 below, it may well produce an infinite number of possible limit surfaces for the same base mesh and, as described under Heuristic 4 below, it would definitely not maintain the rotational symmetries of the mesh. This conjecture has not been tested but, as with Heuristic 1, there seem to be great difficulties with such schemes and, furthermore, the classification mechanism does not admit such schemes. I do not consider them further.

3.2 Heuristics from the Need for a Single Limit Surface

The next two heuristics are based on the desire for a subdivision schemes to produce a single deterministic limit surface, rather than an infinite number of possible limit surfaces. This requires that the limit surface depend solely on the positions and connectivity of the initial base mesh, not on any arbitrary labelling of vertices. These two heuristics exclude those classes which require such an arbitrary labelling.

Heuristic 3. *All vertices at one level of subdivision must map to the same new element type at the next level.*

The term *element* refers to a vertex, face centre, or mid-edge. It is reasonable to require all vertices to be treated identically under refinement because failure to adhere to this heuristic can lead to there being multiple possible limit surfaces for a single base mesh. In these cases, the limit surface will, in general, depend on which particular vertices map to vertices and which do not. This decision must be made at every subdivision step (see Fig. 4) and therefore there is a potentially infinite number of different, equally valid, limit surfaces for any base

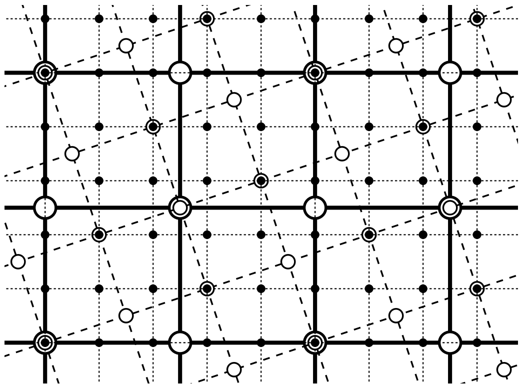

Fig. 4. An example of the arbitrary choices which have to be made in a mixed subdivision schemes. This is $QM\left(\frac{3}{2},\frac{1}{2}\right)$ with the rotation direction alternating on alternate subdivision steps. At the first level of subdivision, half of the vertices map to vertices and the other half map to face centres. At the next level of subdivision half of those vertices map to face centres, and so on. In the limit, at most one of the original vertices will map to a vertex and the choice of this original vertex is arbitrary. There are at least as many limit surfaces as there are original vertices. The mappings for this subdivision class are: $\mathsf{v} \to \mathsf{v}$ *or* f (half of the vertices will map to vertices, the other half to face centres), $\mathsf{f} \to \mathsf{e}$, $\mathsf{e} \to \mathsf{x}$. In the limit, all original vertices (but one) map to no feature at all in the limit surface as they all follow the mapping sequence $\mathsf{v} \to \mathsf{v} \to \cdots \to \mathsf{v} \to \mathsf{f} \to \mathsf{e} \to \mathsf{x} \to \mathsf{x} \to \cdots$

---

mesh. In the case of a finite base mesh, one vertex will be chosen as the origin and all the other vertices will eventually map to no element. There will thus be as many possible limit surfaces as there are vertices in the base mesh. The particular limit surface which is arrived at thus depends on something more than just the location and connectivity of the base mesh's vertices: this is undesirable. In addition, it is difficult to see how such schemes could be extended to handle extraordinary cases, boundaries, and creases.

This heuristic eliminates all mixed classes because, in mixed classes, some vertices map to vertices and some map to face centres. Therefore all TM, QM and HM classes are unlikely to produce useful subdivision methods.

It might be sensible to extend this heuristic to say that all face centres must map to the same element type and that all mid-edges must map to the same element type. This would eliminate most of the TD classes (all except those for which $n + 2m \bmod 6 = 0$, see Table 3 and Appendix A for the detailed calculations of these restrictions). However, while I am convinced that this extension to face centres and mid-edges is sensible, I find it difficult to see how the above argument regarding multiple limit surfaces can be extended to these cases and, furthermore, all classes which would be excluded by such an extension are excluded by the next heuristic anyway.

Heuristic 4. *All rotational symmetries should be maintained under refinement.*

The requirement is that centres of k-fold rotational symmetry (k-centres) at one refinement level have k-fold rotational symmetry at the next level. k-centres may, of course, become centres of higher rotational symmetry provided that the higher symmetry preserves k-fold symmetry. This heuristic seems reasonable because a loss of rotational symmetry leads to multiple possible limit surfaces from the same source mesh. Consider, for example, a vertex in a triangular mesh (6-fold rotational symmetry) which maps to a face centre (3-fold rotational symmetry) under subdivision. There are two possible ways in which this could happen. In simple terms, the vertex maps either to an up-pointing triangle or to a down-pointing triangle. The decision as to which vertices map in which way must be taken at each subdivision step. Even a finite triangular base mesh, with no extraordinary vertices, will thus have infinitely many possible limit surfaces. As with Heuristic 3, the limit surface thus depends on something other than just the location of vertices and the connectivity of the mesh. This is undesirable.

Failing to preserve rotational symmetry also makes it difficult to extend a scheme to handle irregular cases. A particular example of this is considered by Dodgson *et al.* [28] where the $TD(1,1)$ class is explored and a particular $TD(1,1)$ scheme demonstrated; both the particular scheme and the class as a whole are shown to have severe problems. I conjecture that similar problems with irregular cases will arise in any scheme which fails to preserve rotational symmetry. A proof of this conjecture is beyond the scope of this paper because the "multiple limit surface" argument, above, is sufficient justification for this heuristic.

Ivrissimtzis *et al.* [2] suggest that symmetry considerations would be an alternative way to approach the classification problem and it is clear that symmetry considerations are important in subdivision. Han explicitly uses symmetry considerations in his alternative classification mechanism for QP and TP subdivision schemes [26].

The centres of rotational symmetry are the vertices, face centres, and mid-edges of the lattice. I will denote these elements as v, f, and e respectively. The rotational symmetry of each element is shown in Table 1(a).

I use $\rightarrow$ to indicate a mapping of an element from one level of refinement to the next and, in particular, $k \rightarrow k'$ to indicate a mapping from k-fold rotational symmetry to k'-fold rotational symmetry. Under this heuristic, allowable symmetry mappings between values of k and k' are, for Q, $2 \rightarrow 2$, $4 \rightarrow 4$, and $2 \rightarrow 4$; for T and H, $2 \rightarrow 2$, $3 \rightarrow 3$, $6 \rightarrow 6$, $2 \rightarrow 6$, and $3 \rightarrow 6$. Note that $2 \rightarrow 3$ is not allowed because a 3-centre is not also a 2-centre. The mappings in Table 1(b) are thus the only ones which are permitted.

From this we see that triangular dual (TD) classes are not allowed because they map vertices to face centres. Alexa's concentration on the primal classes (TP, $\mathsf{v} \rightarrow \mathsf{v}$) for triangular subdivision is therefore vindicated as neither dual nor mixed schemes are useful in the triangular case.

Table 1. (a) The rotational symmetries of the different elements. (b) The allowable mappings under the restrictions of Heuristic 4.

(a)

Element	Q	T	H
vertex (v)	4	6	3
face centre (f)	4	3	6
mid-edge (e)	2	2	2

(b)

Q	T	H
v $\rightarrow$ v	v $\rightarrow$ v	v $\rightarrow$ v
v $\rightarrow$ f		v $\rightarrow$ f
f $\rightarrow$ v	f $\rightarrow$ v	
f $\rightarrow$ f	f $\rightarrow$ f	f $\rightarrow$ f
e $\rightarrow$ v	e $\rightarrow$ v	
e $\rightarrow$ f		e $\rightarrow$ f
e $\rightarrow$ e	e $\rightarrow$ e	e $\rightarrow$ e

We can also see that any hexagonal scheme which maps face centres to vertices is not allowed, which excludes some of the hexagonal primal HP classes (those for which $(n-m) \bmod 3 = 0$, see Table 4). Thus, of the hexagonal classes, only HD classes and a subset of HP classes are considered useful.

For triangular classes, the only cases in which edges do not map to an appropriate element are already excluded by considering those cases where vertices or face centres do not map to an appropriate element. Therefore it is a moot point whether we need consider the mapping of the rotational symmetries of mid-edges as they are never called into play as a criterion for exclusion. For hexagonal classes, it is possible for an HP class to have v $\rightarrow$ v and f $\rightarrow$ v but to have mid-edges mapping to points with no rotational symmetry (see Table 4). I conjecture that these should also be excluded.

3.3 The Limitations of the Primal/Dual Notation

Details of how the above mappings are calculated can be found in Tables 2–4 and Appendix A. The fact that the calculations for HD and TP and for HP and TD are not exact duals of one another shows up a subtle bias in the classification. The classification is vertex-centric: it explicitly tells us whether a vertex maps to a vertex or a face centre. Arguably of equal significance is whether a face centre maps to a vertex or a face centre. Fortunately this information can be derived directly from the notation. There are four cases:

vv	vertex preserving	v $\rightarrow$ v, f $\rightarrow$ v
ff	face preserving	v $\rightarrow$ f, f $\rightarrow$ f
vf	preserves both	v $\rightarrow$ v, f $\rightarrow$ f
fv	preserves neither	v $\rightarrow$ f, f $\rightarrow$ v

The notation, ab, at left above is shorthand for v $\rightarrow a$, f $\rightarrow b$. This provides a more explicit representation of the mappings which occur than does the simple P and D labelling used by Ivrissimtzis *et al.* [2]. Note that fv is something of a

Table 2. Calculation of the vfe coding for the quadrilateral QP and QD classes. Details of the derivation of these formulæ can be found in Appendix A.

$QP(n,m) \Rightarrow \mathsf{v} \rightarrow \mathsf{v}$

$$(n-m) \bmod 2 = 0 \Rightarrow \mathsf{f} \rightarrow \mathsf{v}$$
$$(n-m) \bmod 2 = 1 \Rightarrow \mathsf{f} \rightarrow \mathsf{f}$$

$$n \bmod 2 = m \bmod 2 = 0 \Rightarrow \mathsf{e} \rightarrow \mathsf{v}$$
$$n \bmod 2 = m \bmod 2 = 1 \Rightarrow \mathsf{e} \rightarrow \mathsf{f}$$
$$n \bmod 2 \neq m \bmod 2 \Rightarrow \mathsf{e} \rightarrow \mathsf{e}$$

Possible scheme types are: vvv, vvf, and vfe.

$QD(n,m) \Rightarrow \mathsf{v} \rightarrow \mathsf{f}$

$$(n-m) \bmod 2 = 0 \Rightarrow \mathsf{f} \rightarrow \mathsf{f}$$
$$(n-m) \bmod 2 = 1 \Rightarrow \mathsf{f} \rightarrow \mathsf{v}$$

$$n \bmod 2 = m \bmod 2 = 0 \Rightarrow \mathsf{e} \rightarrow \mathsf{f}$$
$$n \bmod 2 = m \bmod 2 = 1 \Rightarrow \mathsf{e} \rightarrow \mathsf{v}$$
$$n \bmod 2 \neq m \bmod 2 \Rightarrow \mathsf{e} \rightarrow \mathsf{e}$$

Possible scheme types are: fff, ffv, and fve.

---

special case because a subdivision scheme which is of type fv is of type vf if one considers two steps of subdivision.

Heuristic 4 restricts us to eight useful classes of subdivision scheme. A Q scheme can be any of vv, ff, vf, or fv; while a T scheme can only be vv or vf; and an H scheme can only be ff or vf.

In addition to vertices and face centres, the mappings of mid-edges can also be considered, for completeness. Tables 2–4 show how the mappings can be derived directly from the notation. We see that there are a limited set of valid mappings. I use the notation $\mathsf{vfe} \rightarrow abc$ to indicate $\mathsf{v} \rightarrow a$, $\mathsf{f} \rightarrow b$, and $\mathsf{e} \rightarrow c$, extending the notation above to include edge mappings. Where context is clear I use just abc to indicate the same thing. Note that TD classes allow the possibility that an edge can map to a point with no rotational symmetry (indicated by x) and that half of the face centres can map to face centres while the other half map to vertices (indicated by $\frac{\mathsf{f}}{\mathsf{v}}$). These possibilities are a consequence of allowing the $\mathsf{v} \rightarrow \mathsf{f}$ mapping which reduces a 6-centre to a 3-centre, and provides further justification for Heuristic 4. A similar observation about edges mapping to points with no rotational symmetry can be made about some of the HP classes.

Fig. 5 illustrates the low arity classes. It shows at least one class of each of the mapping types for QP, QD and TP.

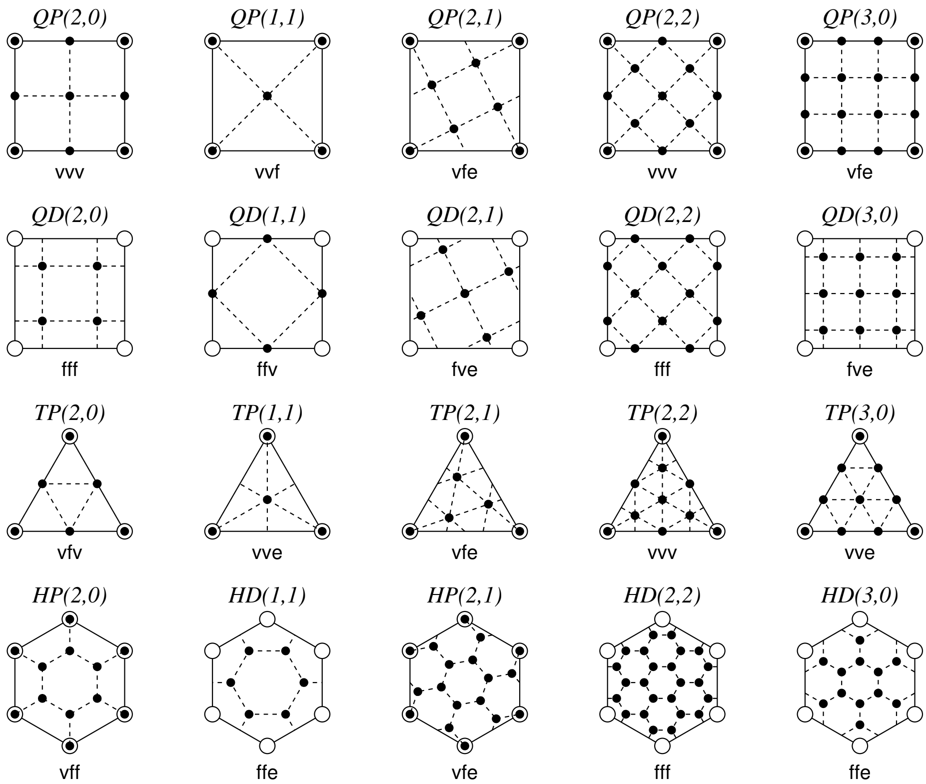

Fig. 5. Illustrations of the low arity QP, QD, TP and H classes. Open circles are source vertices; black dots are subdivided vertices. The solid lines are the source mesh; the dashed lines are the subdivided mesh. The $(2, 1)$ schemes have been included for completeness, although excluded by Heuristic 5.

Table 3. Calculation of the vfe coding for the triangular TP and TD classes. Details of the derivation of these formulæ can be found in Appendix A.

$TP(n,m) \Rightarrow \mathsf{v} \rightarrow \mathsf{v}$		
	$(n-m) \bmod 3 = 0$	$\Rightarrow \mathsf{f} \rightarrow \mathsf{v}$
	otherwise	$\Rightarrow \mathsf{f} \rightarrow \mathsf{f}$
	$n \bmod 2 = m \bmod 2 = 0$	$\Rightarrow \mathsf{e} \rightarrow \mathsf{v}$
	otherwise	$\Rightarrow \mathsf{e} \rightarrow \mathsf{e}$
	Possible scheme types are: vvv, vve, vfv, and vfe.	
$TD(n,m) \Rightarrow \mathsf{v} \rightarrow \mathsf{f}$		
	$(n-m) \bmod 3 = 0$	$\Rightarrow \mathsf{f} \rightarrow \mathsf{f}$
	otherwise	$\Rightarrow \mathsf{f} \rightarrow \genfrac{}{}{0pt}{}{\mathsf{f}}{\mathsf{v}}$
	$n \bmod 2 = m \bmod 2 = 0$	$\Rightarrow \mathsf{e} \rightarrow \mathsf{f}$
	otherwise	$\Rightarrow \mathsf{e} \rightarrow \mathsf{x}$
	Possible scheme types are: fff, ffx, $\mathsf{f}\genfrac{}{}{0pt}{}{\mathsf{f}}{\mathsf{v}}\mathsf{f}$, and $\mathsf{f}\genfrac{}{}{0pt}{}{\mathsf{f}}{\mathsf{v}}\mathsf{x}$.	

3.4 Heuristics Based on Observation of Current Practice

While the previous two heuristics are based on a desire to have a single deterministic limit surface, the following heuristics are much less clear-cut and I therefore address their limitations as well as their merits in the discussion.

Heuristic 5. *Allow only schemes which align the mesh at one level of refinement with the mesh at some higher level of refinement.*

This heuristic was explored by Alexa [1] for the TP classes. It says that a scheme needs to produce a mesh which is in the same rotational orientation as the base mesh after a finite number of steps. For all three types of base mesh, this heuristic permits only $(n,0)$ and (n,n) classes. Alexa [1] proves this for T classes, so it is true for H classes by geometric duality. It is also true for Q classes because it is true by inspection for $(n,0)$ and, for $(n,m),\ m>0$ it requires:

$$\tan\frac{2\pi}{p} \in \mathbb{Q},\ p \in \mathbb{Z}^+,\ 0 < \frac{2\pi}{p} \le \frac{\pi}{4}$$

whose only solution [29] is $p=8$ and therefore $m=n$. In Han's classification of TP and QP schemes [26], his symmetry conditions force this heuristic to be true and his Theorem 2 proves the equivalent of this restriction to $(n,0)$ and (n,n) classes.

This heuristic seems reasonable because the base mesh is often constructed with important linear features of the object aligned with the mesh, so rotating away from this alignment is a bad thing. Of course, the (n,n) classes also rotate away from the desired alignment, but they do it symmetrically and, after two subdivision steps, they are realigned.

Table 4. Calculation of the vfe coding for the hexagonal HP and HD classes. Details of the derivation of these formulæ can be found in Appendix A.

$HP(n, m) \Rightarrow \mathsf{v} \rightarrow \mathsf{v}$

$$\mathsf{v}_\bigtriangledown \rightarrow \mathsf{v}_\bigtriangleup \Rightarrow c = 1$$
$$\mathsf{v}_\bigtriangledown \rightarrow \mathsf{v}_\bigtriangledown \Rightarrow c = 2$$

$(n - m) \bmod 3 = 0$	$\Rightarrow$ f $\rightarrow$ v
$(n - m) \bmod 3 = 3 - c$	$\Rightarrow$ f $\rightarrow$ f
$(n - m) \bmod 3 = c$	$\Rightarrow$ HM, not HP
$(n - m) \bmod 3 = 0$	and
$n \bmod 2 = m \bmod 2 = 0$	$\Rightarrow$ e $\rightarrow$ v
otherwise	$\Rightarrow$ e $\rightarrow$ x
$(n - m) \bmod 3 = 3 - c$	and
$n \bmod 2 = m \bmod 2 = 0$	$\Rightarrow$ e $\rightarrow$ f
otherwise	$\Rightarrow$ e $\rightarrow$ e

Possible scheme types are: vvv, vvx, vff, and vfe.

$HD(n, m) \Rightarrow \mathsf{v} \rightarrow \mathsf{f}$

$(n - m) \bmod 3 = 0$	$\Rightarrow$ f $\rightarrow$ f
otherwise	$\Rightarrow$ HM, not HD
$n \bmod 2 = m \bmod 2 = 0$	$\Rightarrow$ e $\rightarrow$ f
otherwise	$\Rightarrow$ e $\rightarrow$ e

Possible scheme types are: fff and ffe.

However, it is arguable that this heuristic is not strictly necessary. In particular, it is always possible to get the subdivision meshes to realign after every two subdivision steps by performing the rotation one way on even numbered steps and the opposite way on odd numbered steps (an example can be seen in Fig. 4). One way to check the validity of the heuristic would be to perform an investigation (similar to that undertaken for $TD(1, 1)$ [28]) on either of the lowest arity classes which are excluded by this heuristic: $QP(2, 1)$ or $QD(2, 1)$. $QP(2, 1)$ is specifically mentioned by Sloan [3] as useful in the context of numerical integration and Ivrissimtzis *et al.* [30] have recently undertaken an initial investigation of $QP(2, 1)$ schemes. While they do produce a valid subdivision scheme, it is unclear whether it is of practical use.

Heuristic 6. *Triangular and quadrilateral schemes are generally useful but hexagonal schemes are more limited in their applications.*

As mentioned above, it is frequently useful to have important linear features in the model, such as edges, run along an edge in the base mesh in order to preserve the linear feature from one level of subdivision to the next. Hexagonal meshes do not have any straight edges which will run between multiple polygons. This would seem to limit the applicability of hexagonal schemes because they are not useful for objects in which such linear features need to be preserved. However, Claes *et al.* [14] claim that this is one of the advantages of hexagonal schemes: that they can be used situations where one does not want linear features to be preserved. Furthermore, hexagonal dual schemes are useful as the dual of triangular primal schemes [21].

Heuristic 7. *Low arity is preferable to high arity.*

Low arity has one key advantage over high arity: it provides a smaller increase in the number of vertices, which has the desirable effect of allowing for many levels of resolution close to one another. This is one of Kobbelt's [13] justifications for the usefulness of the $\sqrt{3}$ scheme.

Low arity is therefore important. The question then arises, what is the maximum arity that is worth considering. There seems to have been no serious investigation of any class with arity higher than three. For the purposes of this paper, I consider classes of arity less than four. Four is a somewhat arbitrary cut-off point and I make only one, weak, claim for it to be the cut-off, rather than any other value, which is that any arity two (binary) scheme also describes an arity four scheme by simply taking two subdivision steps of the arity two scheme. While an arity four scheme offers greater freedom than that offered by an arity two scheme in terms of choice of coefficients, it is unclear that there would be significant advantage in providing this greater freedom as it comes at the cost of reducing the number of levels of resolution available to the users.

Between arity three and arity four lie the T and H classes of arity $(2,2)$ $(\equiv \sqrt{12})$ and the Q classes of arity $(3,1)$ $(\equiv \sqrt{10})$ and $(3,2)$ $(\equiv \sqrt{13})$. The latter two classes would be excluded by Heuristic 5 but $TP(2,2)$ and $HD(2,2)$ would not be excluded by that heuristic and may be interesting as they are the lowest arity classes with mapping types vvv (triangular) and fff (hexagonal).

It is arguable that we should consider nothing higher than arity three; this would exclude the T and H classes of arity $(2,2)$ but not the $Q(2,2)$ classes $(\equiv \sqrt{8})$. As intimated the start of Sect. 3, it has been suggested that nothing higher than arity two is worth considering, which would exclude the ternary classes (arity $(3,0)$) as well the $Q(2,2)$ classes. Recent work [17, 18, 25] appears to contradict this extreme view and ternary classes certainly allow a range of different behaviour to that permitted by binary classes.

In contradiction of this estimate that arity four is some sort of rough cut-off point, consider the work of Maillot and Stam [31], who provide subdivision of arbitrary integer arity. Their work, however, simply does a single step of subdivision, of appropriate arity, to get from the base mesh to the final mesh, which is not quite in the spirit of subdivision.

Table 5. The low arity classes which may be useful. They are listed in order of increasing arity within the four classifications QP, QD, TP, H. I have included some which are excluded by later heuristics and the right hand column shows which heuristics would cause them to be excluded. Each class is subdivided into approximating and interpolating sub-classes. Interpolating versions of dual schemes are difficult to construct (Heuristic 8) and have therefore been omitted. Sub-classes which have been investigated in the literature are given their common names and an appropriate citation. Those which have not, to my knowledge, been investigated are given a descriptive name in square brackets.

Class	vfe →	Example Schemes Approximating	 Interpolating	Excluded by
$QP(1,1)$	vvf	Velho [11] P&S [10]	interpolating $\sqrt{2}$ [22, 23]	
$QP(2,0)$	vvv	Catmull-Clark [6]	Kobbelt [16]	
$QP(2,1)$	vfe	$\sqrt{5}$ [30]	[*interpolating* $\sqrt{5}$]	Heuristic 5
$QP(2,2)$	vvv	[$\sqrt{8}$]	[*interpolating* $\sqrt{8}$]	
$QP(3,0)$	vfe	[*ternary*]	[*interpolating ternary*]	
$QD(1,1)$	ffv	simplest [9]	—	
$QD(2,0)$	fff	Doo-Sabin [7]	—	
$QD(2,1)$	fve	[*dual* $\sqrt{5}$]	—	Heuristic 5
$QD(2,2)$	fff	[*dual* $\sqrt{8}$]	—	
$QD(3,0)$	fve	[*dual ternary*]	—	
$TP(1,1)$	vve	$\sqrt{3}$ [13]	interpolatory $\sqrt{3}$ [24]	
$TP(2,0)$	vfv	Loop [8]	butterfly [15]	
$TP(2,1)$	vfe	[$\sqrt{7}$]	[*interpolating* $\sqrt{7}$]	Heuristic 5
$TP(3,0)$	vve	Loop ternary [17]	interpolating ternary [18]	
$TP(2,2)$	vvv	[$\sqrt{12}$]	[*interpolating* $\sqrt{12}$]	Heuristic 7(?)
$HD(1,1)$	ffe	hexagon-by-three [14]	—	Heuristic 6
$HP(2,0)$	vff	hex binary [32](?)	[*interpolating hex binary*]	Heuristic 6
$HP(2,1)$	vfe	[*hex* $\sqrt{7}$]	[*interpolating hex* $\sqrt{7}$]	Heuristics 5 and 6
$HD(3,0)$	ffe	[*hex ternary*]	—	Heuristic 6
$HD(2,2)$	fff	[*hex dual* $\sqrt{12}$]	—	Heuristics 6 and 7(?)

Heuristic 8. *Interpolating schemes should be primal.*

All classes can accommodate approximating schemes. Any class with the v → v mapping can also accommodate interpolating schemes. Classes with the v → f mapping are also able to produce interpolating schemes but the derivations required are complicated and it is not clear that the advantages outweigh the complications.

4 Discussion

Taking all these heuristics into account, the arites which will most reward further investigation are $(1,1)$, $(2,0)$, $(3,0)$ and $(2,2)$, producing twelve subdivision classes (eight Q, four T) or eleven if we discount the $TP(2,2)$ class with the rather high arity $\sqrt{12}$. Including the equivalent H classes would add three or four classes to be considered (depending on whether or not one includes $HD(2,2)$). Table 5 lists the low arity classes classes, along with the name of the most well-known published schemes in each class. I have included the $(2,1)$ classes (excluded by Heuristic 5) for completeness because it may be that something useful could be done with them. Fig. 5 shows the layout of a single refinement step for each. Table 5 can be considered a much extended version of Zorin and Schröder's [20] Table 1. It is worth noting that, in addition to the schemes named in Table 5, Zorin and Schröder [20] have developed a whole family of $QP(2,0)$ and $QD(2,0)$ schemes and Oswald and Schröder [21] a whole family of $TP(1,1)$ and $HD(1,1)$ schemes, all based on up-sampling followed by repeated averaging.

The classification allows description of a wide range of possible subdivision schemes ranging from those which are currently used through those which may be useful to those which are almost certainly unusable. The heuristics are a mechanism for paring away the unusable classes in order to clearly identify the useful ones. While the classification system is a clean mathematical construct, the heuristics are less well-defined. The first four heuristics have strong justifications, but the latter four are open to contradiction as demonstrated in the discussion following each heuristic. Note that Ivrissimtzis *et al.* [2] implicitely assume the first two heuristics, while Han [26] assumes the first six. This paper indicates that, in contrast to both of those assumptions, the first *four* are reasonably straightforward to justify. One useful next step would be to ascertain whether there are formal mathematical proofs which either support or shatter each heuristic.

In addition there are open questions pertaining to classes which are identified as useful by the heuristics but which have not yet been investigated:

- Is there any advantage to be gained from using a quadrilateral ternary ($Q(3,0)$) scheme? (c.f. Hassan's [25] univariate ternary scheme and the triangular ternary schemes investigated by Loop [17] and Dodgson *et al.* [18]).
- Is there any advantage in developing a $TP(2,2)$ scheme? $TP(2,2)$ is the lowest arity triangular class where all three element types map to vertices (i.e. it is of mapping type vvv). By contrast, the simplest quadrilateral class with this mapping is the thoroughly investigated $QP(2,0)$ class. However, even the simplest, useful $TP(2,2)$ scheme would require a vertex to have influence outside its 1-ring, making it difficult to extend to extraordinary cases, boundaries, and creases, so it may have little, if any, advantage.
- Are there useful interpolating $QP(1,1)$ and $TP(3,0)$ schemes? While Ivrissimtzis *et al.* [22, 23] have calculated appropriate mask coefficients for the $QP(1,1)$ class and Dodgson *et al.* [18] have undertaken initial work on $TP(3,0)$, it remains to perform detailed analysis and to modify the schemes to handle the extraordinary cases, boundaries, and creases.

It is possible to add further heuristics to the list relating to details further down the classification hierarchy (Sect. 1). As an example, the next heuristic which I would propose is the rather obvious:

Heuristic 9. *A small footprint is preferable.*

A smaller footprint makes for more efficient calculation and is easier to modify to handle the extraordinary cases. A larger footprint gives more freedom in choice of coefficients. Loop's motivation for investigating a ternary version ($TP(3,0)$) [17] of his binary scheme ($TP(2,0)$) [8] was that the ternary version gave more degrees of freedom. As a second example, the higher degree $QP(2,0)$ and $QD(2,0)$ schemes generated by Zorin and Schröder [20] have large footprints and clearly require more calculation than the lower degree schemes which seems to be a contra-indication. However, the mechanism of repeated averaging which they use provides a straightforward way of handling the extraordinary cases at the expense of losing the extra freedoms gained by having a larger footprint and at the expense of severe distortion around extraordinary points.

5 Conclusion

By applying heuristics to the classification, I conclude that the most useful linear, stationary subdivision classes have been investigated and schemes developed for them. There is some scope for further work, principally in looking at ternary subdivision [17, 18]. However the future development of new subdivision schemes seem to lie elsewhere, for example in the development of non-linear or non-stationary versions of schemes for classes which have already been investigated [33] or in combining schemes from more than one class into a single coherent mechanism [10, 34].

Acknowledgements

This work has been supported in part by the European Union, under the ægis of the MINGLE project (HPRN–CT–1999–00117). Thanks to Malcolm Sabin and Nira Dyn for interesting discussions, to the referees of the first Symposium on Geometry Processing who commented on an earlier version of this work and made many useful suggestions, and to the referees of this conference for their helpful comments.

References

1. Alexa, M.: Refinement operators for triangle meshes. Computer Aided Geometric Design **19** (2002) 169–172
2. Ivrissimtzis, I.P., Dodgson, N.A., Sabin, M.A.: A generative classification of mesh refinement rules with lattice transformations. Computer Aided Geometric Design **22** (2004) 99–109
3. Sloan, I.H., Joe, S.: Lattice methods for multiple integration. Oxford Science Publications. Oxford (1994)
4. Warren, J., Weimer, H.: Subdivision Methods for Geometric Design. Morgan Kaufmann (2001)

5. Martin, G.E.: Transformation Geometry. Springer-Verlag, New York (1982)
6. Catmull, E., Clark, J.: Recursively generated B-spline surfaces on arbitrary topological meshes. Computer Aided Design **10** (1978) 350–355
7. Doo, D., Sabin, M.: Behaviour of recursive division surfaces near extraordinary points. Computer-Aided Design **10** (1978) 356–360
8. Loop, C.T.: Smooth subdivision surfaces based on triangles. Master's thesis, University of Utah, Department of Mathematics (1987)
9. Peters, J., Reif, U.: The simplest subdivision scheme for smoothing polyhedra. ACM Transactions on Graphics **16** (1997) 420–431
10. Peters, J., Shiue, L.J.: Combining 4- and 3-direction subdivision. ACM Transactions on Graphics **23** (2004) 980–1003
11. Velho, L.: Quasi 4-8 subdivision. Computer Aided Geometric Design **18** (2001) 299–379
12. Velho, L., Zorin, D.: 4-8 subdivision. Computer Aided Geometric Design **18** (2001) 397–427
13. Kobbelt, L.: $\sqrt{3}$-Subdivision. In: SIGGRAPH 2000 Conference Proceedings. (2000) 103–112
14. Claes, J., Beets, K., van Reeth, F.: A corner-cutting scheme for hexagonal subdivision surfaces. In: Proceedings of Shape Modelling International. (2002) 13–20
15. Dyn, N., Levin, D., Gregory, J.A.: A butterfly subdivision scheme for surface interpolation with tension control. ACM Transactions on Graphics **9** (1990) 160–169
16. Kobbelt, L.: Interpolatory subdivision on open quadrilateral nets with arbitrary topology. Computer Graphics Forum **15** (1996) 409–420
17. Loop, C.T.: Smooth ternary subdivision of triangular meshes. In Cohen, A., Merrien, J.L., Schumaker, L.L., eds.: Curve and Surface Fitting: Saint-Malo 2002, Nashboro Press (2003) 295–302
18. Dodgson, N.A., Sabin, M.A., Barthe, L., Hassan, M.F.: Towards a ternary interpolating subdivision scheme for the triangular mesh. University of Cambridge Computer Laboratory Technical Report No. 539 (2002)
19. Sabin, M.A.: Eigenanalysis and artifacts of subdivision curves and surfaces. In Iske, A., Quak, E., Floater, M.S., eds.: Tutorials on Multiresolution in Geometric Modelling. Springer (2002) 69–92
20. Zorin, D., Schröder, P.: A unified framework for primal/dual quadrilateral subdivision schemes. Computer Aided Geometric Design **18** (2001) 429–454
21. Oswald, P., Schröder, P.: Composite primal/dual $\sqrt{3}$-subdivision schemes. Computer Aided Geometric Design **20** (2003) 135–164
22. Ivrissimtzis, I.P., Dodgson, N.A., Hassan, M.F., Sabin, M.A.: On the geometry of recursive subdivision. International Journal of Shape Modeling **8** (2002) 23–42
23. Ivrissimtzis, I.P., Dodgson, N.A., Sabin, M.A.: Recursive subdivision and hypergeometric functions. In: Proceedings of SMI2002: International Conference on Shape Modelling and Applications, IEEE Press (2002) 145–153
24. Labsik, U., Greiner, G.: Interpolatory $\sqrt{3}$-subdivision. Computer Graphics Forum **19** (2000) 131–138
25. Hassan, M.F., Ivrissimtzis, I.P., Dodgson, N.A., Sabin, M.A.: An interpolating 4-point C^2 ternary stationary subdivision scheme. Computer Aided Geometric Design **19** (2002) 1–18
26. Han, B.: Classification and construction of bivariate subdivision schemes. In Cohen, A., Merrien, J.L., Schumaker, L.L., eds.: Curve and Surface Fitting: Saint-Malo 2002, Nashboro Press (2003) 187–197

27. Ostromoukhov, V., Donohue, C., Jodoin, P.M.: Fast hierarchical importance sampling with blue noise properties. ACM Transactions on Graphics **23** (2004) 488–495 Proc. SIGGRAPH 2004.
28. Dodgson, N.A., Ivrissimtzis, I.P., Sabin, M.A.: Characteristics of dual $\sqrt{3}$ subdivision schemes. In Cohen, A., Merrien, J.L., Schumaker, L.L., eds.: Curve and Surface Fitting: Saint-Malo 2002, Nashboro Press (2003) 119–128
29. Niven, I.M.: Irrational Numbers. Carus Mathematical Monographs. Wiley, New York (1956)
30. Ivrissimtzis, I.P., Sabin, M.A., Dodgson, N.A.: $\sqrt{5}$ subdivision. In Dodgson, N.A., Floater, M.S., Sabin, M.A., eds.: Advances in Multiresolution for Geometric Modelling, Springer-Verlag (2005) 285–299
31. Maillot, J., Stam, J.: A unified subdivision scheme for polygonal modeling. Computer Graphics Forum **20** (2001) 471–479
32. Dyn, N., Levin, D., Simoens, J.: Face value subdivision schemes on triangulations by repeated averaging. In Cohen, A., Merrien, J.L., Schumaker, L.L., eds.: Curve and Surface Fitting: Saint-Malo 2002, Nashboro Press (2003) 129–138
33. Morin, G., Warren, J., Weimer, H.: A subdivision scheme for surfaces of revolution. Computer Aided Geometric Design **18** (2001) 483–502
34. Stam, J., Loop, C.: Quad/triangle subdivision. Computer Graphics Forum **22** (2003) 79–85

A Details of the Formulæ in Tables 2–4

A.1 Quadrilateral Mesh

In the coordinate system of the subdivided mesh, vertices are at $(x, y),\ x, y \in \mathbb{Z}$, face centres at $\left(x + \frac{1}{2}, y + \frac{1}{2}\right),\ x, y \in \mathbb{Z}$, and mid-edges at $\left(x + \frac{1}{2}, y\right)$, $\left(x, y + \frac{1}{2}\right)$, $x, y \in \mathbb{Z}$.

For the primal classes, $QP(n, m)$, the origin of the source grid is a source vertex at $(0, 0)$, with an adjacent source vertex at $(n, m),\ n, m \in \mathbb{Z},\ 0 < n,\ 0 \leq m \leq n$.

A source quadrilateral adjacent to the origin has vertices at $(0, 0)$, (n, m), $(-m, n)$, and $(n - m, n + m)$. Its face centre is at the arithmetic mean of these four points: $\left(\frac{n-m}{2}, \frac{n+m}{2}\right)$. This coincides with a vertex of the subdivided mesh if $n - m \bmod 2 = 0$. If the alternative, $n - m \bmod 2 = 1$, is true then a face centre maps to a face centre.

The source edge from $(0, 0)$ to (n, m) has its midpoint at $\left(\frac{n}{2}, \frac{m}{2}\right)$. Therefore, if $n \bmod 2 = m \bmod 2 = 0$ we have $\mathsf{e} \to \mathsf{v}$, if $n \bmod 2 = m \bmod 2 = 1$, we have $\mathsf{e} \to \mathsf{f}$, and otherwise we have $\mathsf{e} \to \mathsf{e}$.

For the dual classes, $QD(n, m)$, everything shifts by $\left(\frac{1}{2}, \frac{1}{2}\right)$. The net result is that we can simply exchange the rôles of face centres and vertices in subdivided mesh in the $QP(n, m)$ case. Thus, $n - m \bmod 2 = 0 \Rightarrow \mathsf{f} \to \mathsf{f}$ and $n - m \bmod 2 = 1 \Rightarrow \mathsf{f} \to \mathsf{v}$ for the QD case and, likewise, $n \bmod 2 = m \bmod 2 = 0 \Rightarrow \mathsf{e} \to \mathsf{f}$; $n \bmod 2 = m \bmod 2 = 1 \Rightarrow \mathsf{e} \to \mathsf{v}$; otherwise $\mathsf{e} \to \mathsf{e}$.

A.2 Triangular Mesh

In the coordinate system of the subdivided mesh, vertices are at $(x, y),\ x, y \in \mathbb{Z}$, face centres at $\left(x + \frac{1}{3}, y + \frac{1}{3}\right),\ \left(x + \frac{2}{3}, y + \frac{2}{3}\right),\ x, y \in \mathbb{Z}$, and mid-edges at $\left(x + \frac{1}{2}, y\right),\ \left(x, y + \frac{1}{2}\right),\ \left(x + \frac{1}{2}, y + \frac{1}{2}\right),\ x, y \in \mathbb{Z}$. Note that there are *two* types of face centre: the centres of up-pointing triangles ($\triangle$) and the centres of down-pointing triangles ($\triangledown$). The ramifications of this are discussed in detail by Ivrissimtzis *et al.* [2]. We will annotate the f notation with a subscript, $\mathsf{f}_\triangle$ and $\mathsf{f}_\triangledown$, where necessary.

For a $TP(n, m)$ class, without loss of generality, we will take the origin of the source grid to be a source vertex at $(0, 0)$, with an adjacent source vertex at $(n, m),\ n, m \in \mathbb{Z},\ 0 < n,\ 0 \le m \le n$, and with an up-pointing triangle to the left of the line as one moves from $(0, 0)$ to (n, m).

The up-pointing source triangle to the left of this line has source vertices at $(0, 0)$, (n, m) and $(-m, n + m)$. The face centre of this source triangle is at the arithmetic mean of these three points: $\left(\frac{n-m}{3}, \frac{n+2m}{3}\right)$. Thus we have three possible mappings:

$$\begin{aligned} n + 2m \bmod 3 = 0 &\Rightarrow \mathsf{f}_\triangle \to \mathsf{v} \quad \mathsf{f}_\triangledown \to \mathsf{v} \\ n + 2m \bmod 3 = 1 &\Rightarrow \mathsf{f}_\triangle \to \mathsf{f}_\triangle\ \mathsf{f}_\triangledown \to \mathsf{f}_\triangledown \\ n + 2m \bmod 3 = 2 &\Rightarrow \mathsf{f}_\triangle \to \mathsf{f}_\triangledown\ \mathsf{f}_\triangledown \to \mathsf{f}_\triangle \end{aligned}$$

It is not clear that there is a need to distinguish between up- and down-pointing triangles and so, in the interests of clarity, Table 3 does not do so. The reader will note, however, that the most widely used triangular schemes (the $TP(2, 0)$ schemes Loop [8] and butterfly [15]) map up-pointing triangles to down-pointing triangles and vice-versa.

The source edge from $(0, 0)$ to (n, m) has its midpoint at $\left(\frac{n}{2}, \frac{m}{2}\right)$. Therefore, if $n \bmod 2 = m \bmod 2 = 0$ we have $\mathsf{e} \to \mathsf{v}$. In all other cases, $\mathsf{e} \to \mathsf{e}$.

For the $TD(n, m)$ classes, the origin of the source grid is a source vertex at the centre of a face. Its coordinates will thus be: $\left(\frac{c}{3}, \frac{c}{3}\right)\ c \in \{1, 2\}$ where $c = 1$ if the face is an up-pointing triangle and $c = 2$ if the face is a down-pointing triangle. Ivrissimtzis *et al.* [2] show that $n, m \in \mathbb{Z}$ in the TD case.

The up-pointing source triangle to the left of the line from the origin to the adjacent source vertex, $\left(n + \frac{c}{3}, m + \frac{c}{3}\right)$, has vertices at $\left(\frac{c}{3}, \frac{c}{3}\right)$, $\left(n + \frac{c}{3}, m + \frac{c}{3}\right)$ and $\left(-m + \frac{c}{3}, n + m + \frac{c}{3}\right)$. The face centre of this source triangle is at the arithmetic mean of these three points: $\left(\frac{n-m}{3} + \frac{c}{3}, \frac{n+2m}{3} + \frac{c}{3}\right)$. Thus we have three possible mappings for each of the values of c. For $c = 1$:

$$\begin{aligned} n + 2m \bmod 3 = 0 &\Rightarrow \mathsf{f}_\triangle \to \mathsf{f}_\triangle\ \mathsf{f}_\triangledown \to \mathsf{f}_\triangle \\ n + 2m \bmod 3 = 1 &\Rightarrow \mathsf{f}_\triangle \to \mathsf{f}_\triangledown\ \mathsf{f}_\triangledown \to \mathsf{v} \\ n + 2m \bmod 3 = 2 &\Rightarrow \mathsf{f}_\triangle \to \mathsf{v} \quad \mathsf{f}_\triangledown \to \mathsf{f}_\triangledown \end{aligned}$$

For $c = 2$:

$$\begin{aligned} n + 2m \bmod 3 = 0 &\Rightarrow \mathsf{f}_\triangle \to \mathsf{f}_\triangledown\ \mathsf{f}_\triangledown \to \mathsf{f}_\triangledown \\ n + 2m \bmod 3 = 1 &\Rightarrow \mathsf{f}_\triangle \to \mathsf{v} \quad \mathsf{f}_\triangledown \to \mathsf{f}_\triangle \\ n + 2m \bmod 3 = 2 &\Rightarrow \mathsf{f}_\triangle \to \mathsf{f}_\triangle\ \mathsf{f}_\triangledown \to \mathsf{v} \end{aligned}$$

In the TD cases, unless $n + 2m \bmod 3 = 0$, then half of the face centres map to face centres and half map to vertices, which is forbidden by Heuristic 3. However, the situation is rather messy as Heuristic 3 excludes only some of the TD classes, providing further evidence that there are deeper things going on than revealed by the simple classification into 'primal' and 'dual'.

The edge from $\left(\frac{c}{3}, \frac{c}{3}\right)$ to $\left(n + \frac{c}{3}, m + \frac{c}{3}\right)$ has its midpoint at $\left(\frac{n}{2} + \frac{c}{3}, \frac{m}{2} + \frac{c}{3}\right)$. Therefore, if $n \bmod 2 = m \bmod 2 = 0$ we have $\mathsf{e} \to \mathsf{f}$. In all other cases, $\mathsf{e} \to \mathsf{x}$, i.e. an edge maps either to a face centre or it maps to no element at all.

Only if both $n + 2m \bmod 3 = 0$ and $n \bmod 2 = m \bmod 2 = 0$ do we get a sensible mapping. Combining these two gives the condition $n + 2m \bmod 6 = 0$ which is mentioned in the discussion of Heuristic 3.

A.3 Hexagonal Mesh

The hexagonal case is somewhat more involved than the triangular case because, in the hexagonal case, we can distinguish two different types of vertex. This means that we must check that both types of vertex map to the same new element type (face or vertex) in order for the class to be either HD or HP. Otherwise, the class is HM.

In the coordinate system of the subdivided mesh, face centres are at (x, y), $x, y \in \mathbb{Z}$, vertices at $\left(x + \frac{1}{3}, y + \frac{1}{3}\right)$, $\left(x + \frac{2}{3}, y + \frac{2}{3}\right)$, $x, y \in \mathbb{Z}$, and mid-edges at $\left(x + \frac{1}{2}, y\right)$, $\left(x, y + \frac{1}{2}\right)$, $\left(x + \frac{1}{2}, y + \frac{1}{2}\right)$, $x, y \in \mathbb{Z}$. We need to annotate the v notation in order to distinguish the two types of vertex. Where necessary, vertices at $\left(x + \frac{1}{3}, y + \frac{1}{3}\right)$, $x, y \in \mathbb{Z}$ will be denoted $\mathsf{v}_\triangle$ and those at $\left(x + \frac{2}{3}, y + \frac{2}{3}\right)$, $x, y \in \mathbb{Z}$, v_∇. $\mathsf{v}_\triangle$ is a Y-shaped vertex while v_∇ is an inverted Y. The orientation of the triangle is the dual of the configuration of the vertex.

In the hexagonal case, the (n, m) notation does not refer to the distance between two adjacent vertices but between two vertices of the same type or, equivalently, between two face centres. This ensures that the hexagonal cases with classification (n, m) are duals of the triangular cases with classification (n, m).

For an $HD(n, m)$ class, without loss of generality, we will take the origin of the source grid to be a source vertex at $(0, 0)$, with the next source vertex of the same type at (n, m), $n, m \in \mathbb{Z}$, $0 < n$, $0 \leq m \leq n$, and with the vertex at the origin being of type v_∇.

The hexagon has source vertices of type v_∇ at $(0, 0)$, (n, m), and $(-m, n+m)$, with intervening vertices of type $\mathsf{v}_\triangle$ at $\left(\frac{2n+m}{3}, \frac{-n+m}{3}\right)$, $\left(\frac{2n-2m}{3}, \frac{2n+4m}{3}\right)$, and $\left(\frac{-n-2m}{3}, \frac{2n+m}{3}\right)$. The face centre of this source hexagon is at the arithmetic mean of these six points: $\left(\frac{n-m}{3}, \frac{n+2m}{3}\right)$. From these, we can determine that $\mathsf{v}_\nabla \to \mathsf{f}$ always (by definition) and that:

$$
\begin{aligned}
n + 2m \bmod 3 = 0 &\Rightarrow \mathsf{v}_\triangle \to \mathsf{f} \quad \mathsf{f} \to \mathsf{f} \\
n + 2m \bmod 3 = 1 &\Rightarrow \mathsf{v}_\triangle \to \mathsf{v}_\nabla \ \mathsf{f} \to \mathsf{v}_\triangle \\
n + 2m \bmod 3 = 2 &\Rightarrow \mathsf{v}_\triangle \to \mathsf{v}_\triangle \ \mathsf{f} \to \mathsf{v}_\nabla
\end{aligned}
$$

Thus, if $n+2m \bmod 3 \neq 0$, we do not have an HD class because vertices of type $\mathsf{v}_\triangle$ do not map to face centres, and therefore we have an HM class. Thus, for all HD classes, $n+2m \bmod 3 = 0$, by definition, and $\mathsf{f} \to \mathsf{f}$.

Analysis of the edges show that there are only two possible edge mappings. If $n \bmod 2 = m \bmod 2 = 0$ we have $\mathsf{e} \to \mathsf{f}$. In all other cases, $\mathsf{e} \to \mathsf{e}$.

For the $HP(n,m)$ classes let us take, as the origin of the source grid, a source vertex of type $\mathsf{v}_\bigtriangledown$ at $\left(\frac{c}{3}, \frac{c}{3}\right)$ $c \in \{1,2\}$ where the value of c determines the type of destination vertex ($\mathsf{v}_\triangle$ or $\mathsf{v}_\bigtriangledown$). The next source vertex of type $\mathsf{v}_\bigtriangledown$ is at $\left(n+\frac{c}{3}, m+\frac{c}{3}\right)$.

The hexagon has source vertices of type $\mathsf{v}_\bigtriangledown$ at $\left(\frac{c}{3}, \frac{c}{3}\right)$, $\left(n+\frac{c}{3}, m+\frac{c}{3}\right)$, and $\left(-m+\frac{c}{3}, n+m+\frac{c}{3}\right)$, with intervening vertices of type $\mathsf{v}_\triangle$ at $\left(\frac{2n+m+c}{3}, \frac{-n+m+c}{3}\right)$, $\left(\frac{2n-2m+c}{3}, \frac{2n+4m+c}{3}\right)$, and $\left(\frac{-n-2m+c}{3}, \frac{2n+m+c}{3}\right)$. The face centre of this source hexagon is at the arithmetic mean of these six points: $\left(\frac{n-m+c}{3}, \frac{n+2m+c}{3}\right)$. By definition, if $c=1$ then $\mathsf{v}_\bigtriangledown \to \mathsf{v}_\triangle$ and if $c=2$ then $\mathsf{v}_\bigtriangledown \to \mathsf{v}_\bigtriangledown$. We need to consider the mappings for $\mathsf{v}_\triangle$ and f for each value of c.

$$\begin{array}{l}
\text{For } c = 1: \\
n+2m \bmod 3 = 0 \Rightarrow \mathsf{v}_\triangle \to \mathsf{v}_\triangle \ \mathsf{f} \to \mathsf{v}_\triangle \\
n+2m \bmod 3 = 1 \Rightarrow \mathsf{v}_\triangle \to \mathsf{f} \quad \mathsf{f} \to \mathsf{v}_\bigtriangledown \\
n+2m \bmod 3 = 2 \Rightarrow \mathsf{v}_\triangle \to \mathsf{v}_\bigtriangledown \ \mathsf{f} \to \mathsf{f}
\end{array}$$

$$\begin{array}{l}
\text{For } c = 2: \\
n+2m \bmod 3 = 0 \Rightarrow \mathsf{v}_\triangle \to \mathsf{v}_\bigtriangledown \ \mathsf{f} \to \mathsf{v}_\bigtriangledown \\
n+2m \bmod 3 = 1 \Rightarrow \mathsf{v}_\triangle \to \mathsf{v}_\triangle \ \mathsf{f} \to \mathsf{f} \\
n+2m \bmod 3 = 2 \Rightarrow \mathsf{v}_\triangle \to \mathsf{f} \quad \mathsf{f} \to \mathsf{v}_\triangle
\end{array}$$

Thus, we have HM classes if $n+2m \bmod 3 = c$ because, in these cases, $\mathsf{v}_\bigtriangledown \to \mathsf{v}$ but $\mathsf{v}_\triangle \to \mathsf{f}$. For HP schemes we can summarise our results as:

$$\begin{array}{ll}
n+2m \bmod 3 = 0 & \Rightarrow \mathsf{v} \to \mathsf{v} \ \mathsf{f} \to \mathsf{v} \\
n+2m \bmod 3 = 3-c & \Rightarrow \mathsf{v} \to \mathsf{v} \ \mathsf{f} \to \mathsf{f} \\
n+2m \bmod 3 = c & \Rightarrow \text{HM not HP}
\end{array}$$

It now remains to determine the edge mappings. There are three types of edge to consider, which can be characterised by one example of each. These are the first three edges round the source hexagon starting at the origin vertex and they are at:

$$\left(\frac{2n+m}{6}+\frac{c}{3}, \frac{-n+m}{6}+\frac{c}{3}\right),\ \left(\frac{5n+m}{6}+\frac{c}{3}, \frac{-n+4m}{6}+\frac{c}{3}\right),\ \left(\frac{5n-2m}{6}+\frac{c}{3}, \frac{2n+7m}{6}+\frac{c}{3}\right).$$

We thus need to know the values of n and m which place these coordinates at destinations vertices, face centres or edges, which means that we need to consider the values of $2n+m \bmod 6$, $5n+m \bmod 6$ and $5n+4m \bmod 6$. Some basic analysis of these shows that the following results hold:

$$\begin{array}{lll} (n-m) \bmod 3 = 0 & \text{and} & \\ n \bmod 2 = m \bmod 2 = 0 & \Rightarrow & \mathsf{e} \rightarrow \mathsf{v} \\ \text{otherwise} & \Rightarrow & \mathsf{e} \rightarrow \mathsf{x} \end{array}$$

$$\begin{array}{lll} (n-m) \bmod 3 = 3-c & \text{and} & \\ n \bmod 2 = m \bmod 2 = 0 & \Rightarrow & \mathsf{e} \rightarrow \mathsf{f} \\ \text{otherwise} & \Rightarrow & \mathsf{e} \rightarrow \mathsf{e} \end{array}$$