Deriving Box-Spline Subdivision Schemes

Neil A. Dodgson[1], Ursula H. Augsdörfer[1],
Thomas J. Cashman[1], and Malcolm A. Sabin[2]

[1] The Computer Laboratory, University of Cambridge
[2] Numerical Geometry Ltd.
(nad10|uha20|tc270|mas33)@cl.cam.ac.uk

Abstract. We describe and demonstrate an arrow notation for deriving box-spline subdivision schemes. We compare it with the z-transform, matrix, and mask convolution methods of deriving the same. We show how the arrow method provides a useful graphical alternative to the three numerical methods. We demonstrate the properties that can be derived easily using the arrow method: mask, stencils, continuity in regular regions, safe extrusion directions. We derive all of the symmetric quadrilateral binary box-spline subdivision schemes with up to eight arrows and all of the symmetric triangular binary box-spline subdivision schemes with up to six arrows. We explain how the arrow notation can be extended to handle ternary schemes. We introduce two new binary dual quadrilateral box-spline schemes and one new $\sqrt{2}$ box-spline scheme. With appropriate extensions to handle extraordinary cases, these could each form the basis for a new subdivision scheme.

Published in *Mathematics of Surfaces XIII*, E. R. Hancock, R. R. Martin, M. A. Sabin (editors), Lecture Notes in Computer Science, Vol. 5654, pp. 106-123, Springer-Verlag, Sep 2009, ISBN 978-3-642-03595-1, ISSN 0302-9743

1 Introduction

For several years, the Cambridge subdivision research team have used an arrow notation that allows easy derivation of the mask (Sect. 3.1), stencils (Sect. 3.2), continuity (Sect. 3.3), and safe extrusion directions (Sect. 3.4) of all box-spline subdivision schemes. It also permits enumeration of all possible box-spline schemes, which has allowed us to generate three new schemes which, to the best of our knowledge, have not yet been investigated.

The arrow notation is equivalent to other mechanisms for specifying box-spline schemes but has the advantage that it is a graphical, rather than numerical, notation, allowing easy visualisation of what is going on.

The notation has arrows of appropriate lengths pointing in the principal directions of the scheme. For binary schemes, in z-transform space [1], each arrow corresponds to a factor of $(1 + z)/2$ in the appropriate direction; for an n-ary scheme, to a factor of $(1 - z^n)/(n(1 - z))$.

We explain the arrow notation and the properties that can be derived from it for univariate (Sect. 2) and bivariate (Sect. 3) binary subdivision schemes. We use the simplest four-direction scheme (Sect. 3.6) as an example to demonstrate the four ways of deriving a scheme's mask: arrows, z-transform, matrix, and mask convolution. We enumerate all possible binary quadrilateral schemes with up to eight arrows (Sect. 3). We then consider the extension of the method to longer arrows representing factors of $1 + z^2$ (Sect. 4), to triangular meshes (Sect. 5), and to ternary schemes (Sect. 6). We conclude with suggestions for further work (Sec. 7).

2 Univariate Binary Schemes

In one-dimension we would represent the cubic box spline as four arrows:

$$\rightarrow\rightarrow\rightarrow\rightarrow$$

which corresponds to $2((1 + z)/2)^4$. This leads to the Laurent polynomial $(1 + 4z + 6z^2 + 4z^3 + z^4)/8$ which is itself the z-transform of the subdivision mask $[1, 4, 6, 4, 1]/8$ which has the two stencils $[1, 6, 1]/8$ and $[4, 4]/8$. For the purposes of this paper, we ignore the constant factor (in this case, one eighth) when it gets in the way of clear exposition, as it is trivial to derive from the fact that each stencil must sum to one.

Graphically, the arrow notation allows us to derive the mask directly by finding a number of distinct combinations (N.B., *not* permutations) of arrows which get us from the origin to each possible point on the number line. Label each arrow individually:

$$\xrightarrow{a}\xrightarrow{b}\xrightarrow{c}\xrightarrow{d}$$

There is one way to get to the origin (use no arrows), four to get to the first position (use any one of the arrows: $\{a, b, c, d\}$), six to get to the second position: $\{ab, ac, ad, bc, bd, cd\}$, four to the third: $\{abc, abd, acd, bcd\}$, and one to the fourth: $\{abcd\}$. This is simple combinatorics and it parallels exactly the derivation of the co-efficients on the polynomial product in the z-transform. The true usefulness of the graphical notation does not become apparent until we consider bivariate schemes.

Continuity can also be determined from the graphical notation. Again, this is only truly useful when we consider bivariate schemes. Each arrow represents an integration step. Each integration represents an increase in continuity by one. You may prefer to think of this as each arrow representing a multiplication by a factor of $(1 + z)/2$, or a single smoothing step [2] in a refine-and-smooth formulation. In terms of the limit basis functions of the scheme, if there are no arrows, then we have an impulse function. One arrow integrates this to a step function, which is a function containing a discontinuity. A second arrow will integrate this to produce a $C0$ function. From here, each extra arrow adds one to the continuity. Thus, in the univariate case, continuity is two fewer than the number of arrows.

Figure 1 lists the first four univariate box-spline subdivision schemes.

Arrows	z-transform	Mask	Continuity
→→	$2\left(\frac{(1+z)}{2}\right)^2$	$\frac{1}{2}[1,2,1]$	$C0$
→→→	$2\left(\frac{(1+z)}{2}\right)^3$	$\frac{1}{4}[1,3,3,1]$	$C1$
→→→→	$2\left(\frac{(1+z)}{2}\right)^4$	$\frac{1}{8}[1,4,6,4,1]$	$C2$
→→→→→	$2\left(\frac{(1+z)}{2}\right)^5$	$\frac{1}{16}[1,5,10,10,5,1]$	$C3$

Fig. 1. The linear, quadratic, cubic and quartic binary univariate box-spline subdivision schemes. It is straightforward to extend this to higher powers of $(1+z)/2$.

Arrows	z-transform	Mask	Continuity
↑↑ →→	$4\left(\frac{1+z_1}{2}\right)^2\left(\frac{1+z_2}{2}\right)^2$	$\frac{1}{4}\begin{bmatrix}1&2&1\\2&4&2\\1&2&1\end{bmatrix}$	$C0$
↑↑↑ →→→	$4\left(\frac{1+z_1}{2}\right)^3\left(\frac{1+z_2}{2}\right)^3$	$\frac{1}{16}\begin{bmatrix}1&3&3&1\\3&9&9&3\\3&9&9&3\\1&3&3&1\end{bmatrix}$	$C1$
↑↑↑↑ →→→→	$4\left(\frac{1+z_1}{2}\right)^4\left(\frac{1+z_2}{2}\right)^4$	$\frac{1}{16}\begin{bmatrix}1&4&6&4&1\\4&16&24&16&4\\6&24&36&24&6\\4&16&24&16&4\\1&4&6&4&1\end{bmatrix}$	$C2$

Fig. 2. The linear, quadratic and cubic tensor product bivariate box-spline subdivision schemes.

3 Bivariate Binary Quadrilateral Schemes

The tensor product schemes are straightforward to calculate and to represent in arrow notation (Fig. 2). While it is possible to have a different number of arrows in the two primary principal directions, it is generally desirable to have the same number in each, because to do otherwise leads to an asymmetry in the subdivision scheme, which would effectively prevent the generalisation of the box spline into a subdivision scheme with extraordinary vertices.

3.1 Deriving the Mask

The mask of a subdivision scheme shows the contribution of a single original vertex to each new, subdivided vertex. To find the mask of a scheme, we need to find all ways to get from the origin to each point in the grid. For the tensor product schemes, this is simply the tensor product of the univariate case, as the two principal directions are orthogonal.

The process is straightforward. Given the set of arrows for the scheme, find where each possible *combination* of arrows takes us, and then count how many combinations end up in each particular location. Take the cubic tensor product bivariate box-spline scheme. If we label each arrow individually, then the process is easy to follow:

$$\begin{array}{l} h \uparrow \\ g \uparrow \\ f \uparrow \\ e \uparrow \xrightarrow{a} \xrightarrow{b} \xrightarrow{c} \xrightarrow{d} \end{array}$$

There is one way to get from the origin to itself: use no arrows. There are four ways to get to the next position across: $\{a, b, c, d\}$. There are sixteen ways to get to the position above that: $\{ae, af, ag, ah, be, bf, bg, bh, ce, cf, cg, ch, de, df, dg, dh\}$, and so on.

For schemes with non-orthogonal arrows, the situation is rather more interesting, and the arrows prove more useful. A detailed non-orthogonal example is given in Sect. 3.6 and Fig. 4.

3.2 Deriving Stencils

The stencils of a subdivision scheme show how to make a new, subdivided vertex from the surrounding original vertices. There are several stencils for any given scheme, each corresponding to a particular type of new vertex.

From the mask, it is straightforward to derive the stencils. For any binary bivariate scheme, there are four stencils, each derived as every second value on every second row. This is illustrated in Fig. 3. For a ternary scheme there would be nine stencils, each derived as every third value on every third row. Other arities have similar rules. Strictly, the mask should be mirror-imaged about its centre before extracting the stencils, but all masks in this paper are mirror-symmetric so this is not necessary.

3.3 Continuity

Calculating the continuity needs some explanation. We need to know what continuity to expect across any edge in the final mesh. Arrows which point *along* an edge cannot contribute to continuity *across* the edge. Therefore, to calculate continuity, find the direction with the maximum number of arrows. Discard those arrows and count the number of remaining arrows. Continuity is two fewer than this number, for the reasons given in Sect. 2. All of the tensor product schemes have the same continuity as their univariate counterparts. Note that we must consider the edges with minimum continuity and so we cannot claim any higher continuity for the scheme as a whole even if there are other directions where fewer arrows would be discarded.

If we extend to trivariate subdivision, for example for Finite Element Meshing, then a similar argument holds. The continuity across boundaries can be

$$\begin{bmatrix} \boxed{1} & 4 & \boxed{6} & 4 & \boxed{1} \\ 4 & 16 & 24 & 16 & 4 \\ \boxed{6} & 24 & \boxed{36} & 24 & \boxed{6} \\ 4 & 16 & 24 & 16 & 4 \\ \boxed{1} & 4 & \boxed{6} & 4 & \boxed{1} \end{bmatrix} \qquad \begin{bmatrix} 1 & \boxed{4} & 6 & \boxed{4} & 1 \\ 4 & 16 & 24 & 16 & 4 \\ 6 & \boxed{24} & 36 & \boxed{24} & 6 \\ 4 & 16 & 24 & 16 & 4 \\ 1 & \boxed{4} & 6 & \boxed{4} & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 6 & 4 & 1 \\ \boxed{4} & 16 & \boxed{24} & 16 & \boxed{4} \\ 6 & 24 & 36 & 24 & 6 \\ \boxed{4} & 16 & \boxed{24} & 16 & \boxed{4} \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 4 & 6 & 4 & 1 \\ 4 & \boxed{16} & 24 & \boxed{16} & 4 \\ 6 & 24 & 36 & 24 & 6 \\ 4 & \boxed{16} & 24 & \boxed{16} & 4 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix}$$

Fig. 3. Deriving the four stencils from the cubic box-spline mask. Top left: vertex, top right: horizontal edge, bottom left: vertical edge, bottom right: face centre. Each should be divided by a factor of 64.

determined by selecting the plane that contains the maximum number of arrows, discarding these arrows, counting the remaining arrows, and subtracting two.

3.4 Safe Extrusion Directions

Lateral artifacts occur in the limiting surface if the original data is extruded in a direction for which the z-transform of the mask does not have a $(1+z)$ factor [3]. The arrow notation quickly allows one to see which are the safe directions: they are the ones in which there is an arrow. For the tensor product schemes, there are only two safe directions. Schemes with diagonal terms (Sect. 3.5) have four safe directions. Triangular schemes (Sect. 5) have three or six safe directions.

3.5 Diagonal Terms

In addition to horizontal and vertical arrows, quadrilateral schemes can have arrows on the 45° diagonals. These correspond to $(1 + z_1 z_2)$ and $(1 + z_2/z_1)$. We consider the horizontal and vertical arrows to be the primary principal directions, with the 45° arrows being the secondary principal directions. It is generally desirable to have the same number in each of the two primary principal directions, as in the tensor product schemes, because to do otherwise leads to an asymmetry in the subdivision scheme. Similarly, it is desirable to have the same number of arrows in each of the two secondary principal directions. However, it is not necessary to have the same number of arrows in the primary and secondary directions. For example, the tensor product schemes have no arrows in the secondary principal directions.

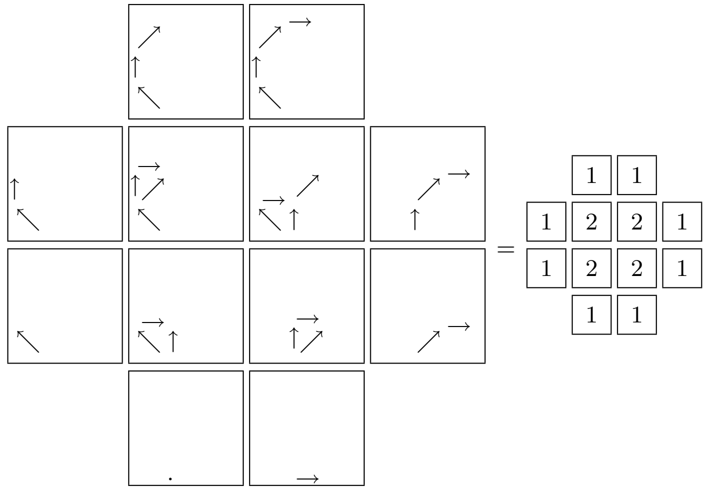
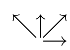


Fig. 4. Using the graphical arrow notation to derive the mask. At left we see all possible paths from the origin to each of the twelve reachable points. At right is a count of the number of paths, which is the mask of the scheme.

3.6 Four-arrow, Four-direction Scheme

The simplest box-spline subdivision scheme that uses diagonal terms is:



To find the mask of this scheme, we need to find all ways to get from the origin to each point in the grid (Fig. 4). Alternatively, we can get the same answer from the z-transform by expanding $(1+z_1)(1+z_2)(1+z_1z_2)(1+z_2/z_1)$:

$$\begin{array}{c} z_1^0 z_2^3 + \; z_1^1 z_2^3 + \\ z_1^{-1} z_2^2 + 2z_1^0 z_2^2 + 2z_1^1 z_2^2 + z_1^2 z_2^2 + \\ z_1^{-1} z_2^1 + 2z_1^0 z_2^1 + 2z_1^1 z_2^1 + z_1^2 z_2^1 + \\ z_1^0 z_2^0 + \; z_1^1 z_2^0 \end{array}$$

Arranging the terms horizontally by increasing exponent on z_1 and vertically by increasing exponent on z_2 produces an array where the coefficients on the terms are the coefficients of the mask.

A third alternative is to use mask convolution. The four simple masks, each of which represents a $(1+z)$ term or an arrow in a principal direction, are convolved to produce the final mask.

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} * \begin{bmatrix} 1\,1 \end{bmatrix} * \begin{bmatrix} 0\,1 \\ 1\,0 \end{bmatrix} * \begin{bmatrix} 1\,0 \\ 0\,1 \end{bmatrix} = \begin{bmatrix} 0\,1\,1\,0 \\ 1\,2\,2\,1 \\ 1\,2\,2\,1 \\ 0\,1\,1\,0 \end{bmatrix}$$

$$\begin{bmatrix} \boxed{0} & 1 & \boxed{1} & 0 \\ 1 & 2 & 2 & 1 \\ \boxed{1} & 2 & \boxed{2} & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & \boxed{1} & 1 & \boxed{0} \\ 1 & 2 & 2 & 1 \\ 1 & \boxed{2} & 2 & \boxed{1} \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 1 & 0 \\ \boxed{1} & 2 & \boxed{2} & 1 \\ 1 & 2 & 2 & 1 \\ \boxed{0} & 1 & \boxed{1} & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & \boxed{2} & 2 & \boxed{1} \\ 1 & 2 & 2 & 1 \\ 0 & \boxed{1} & 1 & \boxed{0} \end{bmatrix}$$

Fig. 5. The four stencils derived from the mask. All four are rotational variants of one another.

The final alternative is to use Peters and Shuie's matrix of directions [4]:

$$\mathcal{A}_{\text{simplest}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Each direction corresponds to one of the arrows in the arrow notation, to one of the terms in the Laurent polynomial (z-transform), and to one of the masks in the mask convolution method.

The four stencils of the scheme can be derived by taking every second row from every second column, in the four possible ways this can be done (Fig. 5).

One of the interesting things about this scheme is that it can be factorised into a $\sqrt{2}$ scheme. One step of that scheme being $\nwarrow\nearrow$ combined with the rotation of 45°, the next step being $\uparrow\!\rightarrow$ with a further rotation of 45°, which realigns the subdivided mesh's primary directions with the original mesh's primary directions. This is the "simplest" subdivision scheme described by Peters and Reif [5].

The mask of $\nwarrow\nearrow$ is $\begin{bmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{bmatrix}$. The two stencils are $\begin{bmatrix} 1 & 1 \end{bmatrix}$ for horizontal edges and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for vertical edges. The mask of $\uparrow\!\rightarrow$ is $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. This is exactly the other simple mask rotated by 45° and contracted by a factor of $\sqrt{2}$. Convolving the two produces the mask of the binary scheme:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} * \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

To derive the continuity of the scheme, consider the direction with the greatest number of arrows. There is one arrow in each of the principal directions, whichever you choose. This leaves three arrows and the continuity is two fewer than that. Therefore the simplest scheme has $C1$-continuity in regular regions. It has four safe extrusion directions: the four principal directions.

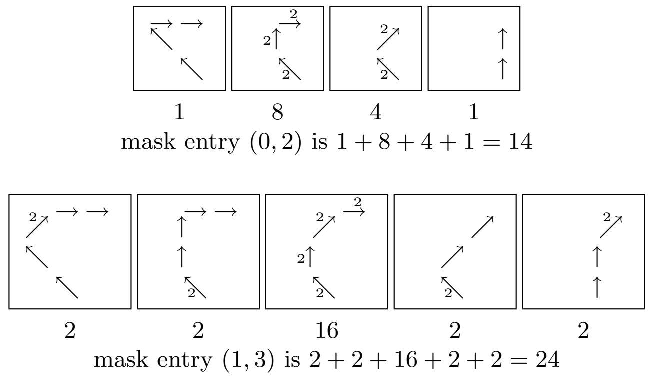
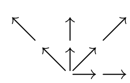


Fig. 6. Two examples of determining the mask entries using the arrow notation, for the eight-arrow, four-direction scheme. Arrows carry an annotation “2” when there are two possible arrows available. The number of possible combinations is shown under each diagram.

In general, a binary box-spline scheme can be factored into a $\sqrt{2}$ scheme if the arrows can be split into two sets, one of which maps onto the other by a rotation of 45° and a dilation of $\sqrt{2}$.

3.7 Eight-arrow, Four-direction Scheme

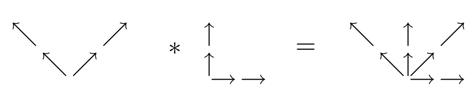
In a similar way, we can evaluate the scheme with two arrows in each of the four principal directions. This is equivalent to two steps of the 4–8 scheme described by Velho [6].



This scheme has continuity $C4$. This is determined by finding the direction with the greatest number of arrows (any one of the four principal directions), counting the number of arrows not in this direction (six) and subtracting two.

The entries in the mask can be determined in any of the ways described above. As an example, consider using the arrows to determine the entries. Figure 6 shows the derivation of two of the entries in the mask by this method. Clearly the more arrows there are, the more complex this procedure becomes and the less useful as a mechanism for deriving the mask entries.

Like the four-arrow, four-direction scheme, this binary scheme is factorisable into a $\sqrt{2}$ scheme:



$$\begin{bmatrix} & & 1 & & \\ & \boxed{2} & & \boxed{2} & \\ 1 & & 4 & & 1 \\ & \boxed{2} & & \boxed{2} & \\ & & 1 & & \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \qquad \begin{bmatrix} & & \boxed{1} & & \\ & 2 & & 2 & \\ \boxed{1} & & \boxed{4} & & \boxed{1} \\ & 2 & & 2 & \\ & & \boxed{1} & & \end{bmatrix} \rightarrow \begin{bmatrix} & 1 & \\ 1 & 4 & 1 \\ & 1 & \end{bmatrix}$$

Fig. 7. The stencils of Velho's 4–8 scheme [6] derived from its mask. There is one stencil for introducing new vertices at face centres (left) and one stencil for moving old vertices (right). Note that values in the mask and stencils must be divided by eight to ensure that the values in each stencil sum to one.

The two $\sqrt{2}$ masks can be convolved to produce the mask of the binary scheme:

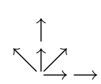
$$\begin{bmatrix} 0&0&1&0&0 \\ 0&2&0&2&0 \\ 1&0&4&0&1 \\ 0&2&0&2&0 \\ 0&0&1&0&0 \end{bmatrix} * \begin{bmatrix} 1&2&1 \\ 2&4&2 \\ 1&2&1 \end{bmatrix} = \begin{bmatrix} 0&0&1&2&1&0&0 \\ 0&2&6&8&6&2&0 \\ 1&6&14&18&14&6&1 \\ 2&8&18&24&18&8&2 \\ 1&6&14&18&14&6&1 \\ 0&2&6&8&6&2&0 \\ 0&0&1&2&1&0&0 \end{bmatrix}$$

You can easily extract Velho's 4–8 stencils from the $\sqrt{2}$ mask (Fig. 7). However, you need to note that there are only two stencils in a $\sqrt{2}$ scheme, compared with four in a binary scheme. Clearly, you could extract four stencils from the binary mask, and thus directly create a binary scheme. This would, however, produce stencils which are large. Large stencils make it more difficult to create mechanisms for the efficient handling of extraordinary vertices, edges and creases in the mesh.

3.8 Six-arrow Schemes

There are two schemes which each have six arrows with at least one in each of the four principal directions, and which each produce smaller stencils than the binary version of Velho's 4–8 scheme.

The first of the two six-arrow schemes is the quadrilateral part of Peters and Shiue's 4–3 scheme [4]. In arrow notation it is:



From this we can see that the scheme is $C2$ in regular regions. It has the mask:

$$\begin{bmatrix} & 1 & 2 & 1 & \\ 1 & 4 & 6 & 4 & 1 \\ 2 & 6 & 8 & 6 & 2 \\ 1 & 4 & 6 & 4 & 1 \\ & 1 & 2 & 1 & \end{bmatrix}$$

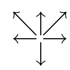
and the four stencils:

$$\begin{bmatrix} & 2 & \\ 2 & 8 & 2 \\ & 2 & \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 6 & 6 \\ 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 6 & 1 \\ 1 & 6 & 1 \end{bmatrix} \qquad \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$$

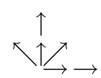
As illustrated in Sect. 3.6, the arrow notation is a straightforward graphical representation of the matrix of directions as used by Peters and Shuie [4]:

$$\mathcal{A}_\triangle = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

This corresponds to the arrow notation:

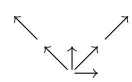


A shift of origin makes no difference to the resulting mask, so this is equivalent to:



We prefer the latter version, where all of the arrows which lie on the same line point in the same direction, as we believe that this makes the notation clearer.

The other six-arrow box-spline is:



which is clearly also $C2$ in regular regions. This has the mask:

$$\begin{bmatrix} & & 1 & 1 & & \\ & 2 & 3 & 3 & 2 & \\ 1 & 3 & 6 & 6 & 3 & 1 \\ 1 & 3 & 6 & 6 & 3 & 1 \\ & 2 & 3 & 3 & 2 & \\ & & 1 & 1 & & \end{bmatrix}$$

and its four stencils are all rotational variants of:

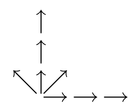
$$\begin{bmatrix} & 1 & \\ 1 & 6 & 3 \\ & 3 & 2 \end{bmatrix}$$

This is a dual binary quadrilateral subdivision scheme, making it most closely related to the quadratic tensor product box spline ($C1$, Doo-Sabin [7, 8]) and the quartic tensor product box spline ($C3$) [9, 10]. This second six-arrow box spline has never been extended to handle extraordinary cases, edges, or creases. We expect that this could be done reasonably easily, given the simplicity of the stencil.

Note that neither of the six-arrow schemes can be factorised into a $\sqrt{2}$ scheme, because neither meets the criteria described at the end of Sect. 3.6.

3.9 More Arrows — Larger Stencils

It is clearly possible to add more arrows. For example, adding two more arrows to either Peters and Shiue's 4–3 scheme or the Doo-Sabin scheme produces an eight-arrow box spline:



This is $C3$. Its mask is:

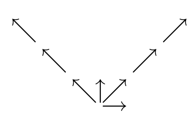
$$\begin{bmatrix} & 1 & 3 & 3 & 1 & \\ 1 & 6 & 13 & 13 & 6 & 1 \\ 3 & 13 & 24 & 24 & 13 & 3 \\ 3 & 13 & 24 & 24 & 13 & 3 \\ 1 & 6 & 13 & 13 & 6 & 1 \\ & 1 & 3 & 3 & 1 & \end{bmatrix}$$

and its four stencils are all rotational variants of:

$$\begin{bmatrix} & 3 & 1 \\ 3 & 24 & 13 \\ 1 & 13 & 6 \end{bmatrix}$$

This is yet another dual binary quadrilateral subdivision scheme. This eight-arrow box spline has never been extended to handle extraordinary cases, edges, or creases.

The final binary quadrilateral scheme with eight arrows is:



This is $C3$ and has a large mask:

$$\begin{bmatrix} & & & 1 & 1 & & & \\ & & 3 & 4 & 4 & 3 & & \\ & 3 & 6 & 12 & 12 & 6 & 3 & \\ 1 & 4 & 12 & 18 & 18 & 12 & 4 & 1 \\ 1 & 4 & 12 & 18 & 18 & 12 & 4 & 1 \\ & 3 & 6 & 12 & 12 & 6 & 3 & \\ & & 3 & 4 & 4 & 3 & & \\ & & & 1 & 1 & & & \end{bmatrix}$$

$$\begin{bmatrix} & \\ 128 & 64 \\ 64 & \end{bmatrix} \quad \begin{bmatrix} & \\ 144 & 48 \\ 48 & 16 \end{bmatrix} \quad \begin{bmatrix} & 16 & \\ 16 & 96 & 48 \\ & 48 & 32 \end{bmatrix} \quad \begin{bmatrix} & 12 & 4 \\ 12 & 96 & 52 \\ 4 & 52 & 24 \end{bmatrix} \quad \begin{bmatrix} 1 & 10 & 5 \\ 10 & 100 & 50 \\ 5 & 50 & 25 \end{bmatrix}$$

Fig. 8. The stencils of the five dual quadrilateral binary box-spline schemes with masks of up to 3×3. All stencils have the common denominator of 256 to allow easy comparison. From left to right: simplest ($C0$) [5], Doo-Sabin ($C1$) [7, 8], six-arrow ($C2$), eight-arrow ($C3$), quartic tensor product (ten-arrow, $C3$) [9, 10].

and a large stencil:

$$\begin{bmatrix} & & 1 & \\ & 6 & 12 & 3 \\ 1 & 12 & 18 & 4 \\ & 3 & 4 & \end{bmatrix}$$

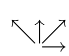
This is another binary dual scheme. In this case, the stencil is so large as to make it difficult to generalise to extraordinary cases.

Consider all of the dual schemes with stencils up to size 3×3. We know, simply by enumerating all possible combinations of arrows, that there are only five of them (ignoring the trivial case which has just two arrows and which does not produce a limit surface). If we use a common weighting factor of $1/256$, then the stencils are as shown in Fig. 8. The largest is the biquartic box-spline, for which extraordinary cases were considered by Qu [9], Zorin and Schröder [10]. It is not clear what, if any, advantage would be gained from the three larger stencils, compared with the two smaller ones. It would be interesting to have all five implemented, and compared against one another, and for mechanisms to be developed to handle extraordinary cases, edges and creases for the six-arrow and eight-arrow schemes.

4 Longer Arrows

It is possible to create schemes with squared or higher terms of z in their z-transform. In the arrow notation these are represented by longer arrows. For example $(1 + z^2)$ is represented by an arrow of twice the length of that representing $(1 + z)$.

The most interesting place where such a binary scheme arises is when we consider a $\sqrt{2}$ scheme with the arrow symbol:

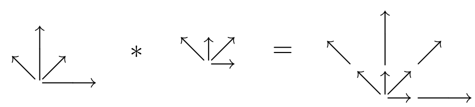


This is the same symbol as for the binary simplest scheme (Sect. 3.6). However, it can also be implemented as a $\sqrt{2}$ scheme in its own right. This is done by deriving two, rather than four, stencils from the mask (Fig. 4). The stencils are read off from the mask at 45°. Rotating them by 45°, the two stencils are:

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \qquad\qquad \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 1 & 1 \end{bmatrix}$$

for new vertices at the nominal centres of the vertical and horizontal edges respectively. This is straightforward to implement and it should be relatively straightforward to generalise to the extraordinary cases, edges and creases.

Taking the convolution of two $\sqrt{2}$ steps, where the second step is rotated by 45° and dilated by $\sqrt{2}$, gives a binary scheme with arrow symbol:



and with z-transform $(1+z_1)(1+z_1^2)(1+z_2)(1+z_2^2)(1+z_1z_2)^2(1+z_2/z_1)^2$. Note the positions of the exponents both inside and outside the parentheses.

The mask of this binary scheme is:

$$\begin{bmatrix} & & 1 & 1 & 1 & 1 & & \\ & 2 & 3 & 5 & 5 & 3 & 2 & \\ 1 & 3 & 8 & 10 & 10 & 8 & 3 & 1 \\ 1 & 5 & 10 & 14 & 14 & 10 & 5 & 1 \\ 1 & 5 & 10 & 14 & 14 & 10 & 5 & 1 \\ 1 & 3 & 8 & 10 & 10 & 8 & 3 & 1 \\ & 2 & 3 & 5 & 5 & 3 & 2 & \\ & & 1 & 1 & 1 & 1 & & \end{bmatrix}$$

Note the four entries of "1" along each of the horizontal and vertical edges. This is the most obvious indication that those double length arrows are doing something different to that observed when only single length arrows are used.

The stencils of the binary scheme are the four rotations of:

$$\begin{bmatrix} & 3 & 5 & 2 \\ 1 & 10 & 14 & 5 \\ 1 & 8 & 10 & 3 \\ & 1 & 1 & \end{bmatrix}$$

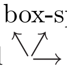
This is a big stencil and, thus, one would not expect to implement it as a binary scheme nor try to extend the binary scheme to extraordinary cases. Instead, as with 4–8, any such extension would be implemented for the $\sqrt{2}$ scheme.

While this scheme is certainly valid there is an interesting question, when determining the continuity, as to how those double length arrows contribute towards continuity. If they contribute as for single length arrows, then the scheme is $C4$ in regular regions. If not, it is almost certainly at least $C2$. Analysis of these double length arrows is best done in the univariate case in a similar way to that done by Dyn [11] and Hassan [12]. Regardless of whether it is $C4$ or $C2$ in regular regions, the scheme would benefit from further investigation and extension to extraordinary cases.

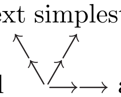
We note, in passing, that it would also be possible to have “knight’s move” arrows. It is unclear what the implications of these would be. Would they provide extra safe extrusion directions? Would they contribute towards higher continuity?

5 Triangular Schemes

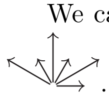
Triangular schemes have six principal directions, three primary and three secondary, but are otherwise handled in much the same way as for quadrilaterals. Two of the primary directions are $(1 + z_1)$ and $(1 + z_2)$. It is a matter of convention whether the third primary direction should be treated as $(1 + z_1 z_2)$ or $(1 + z_2/z_1)$, depending on whether you prefer the positive z_1 and z_2 axes to be separated by 60° or 120°. Note that $(1 + z_2/z_1)$ can be shifted to $(z_1 + z_2)$ if you prefer only non-negative powers of z. The shift has no effect on the resulting mask.

The simplest box-spline triangular scheme, linear interpolation, is $C0$, has the arrow symbol  and the mask:

$$\begin{array}{ccccc} & 1 & & 1 & \\ 1 & & 2 & & 1 \\ & 1 & & 1 & \end{array}$$

The next simplest, Loop subdivision [13], is $C2$ in regular regions, has the arrow symbol  and the mask:

$$\begin{array}{ccccccccc} & & 1 & & 2 & & 1 & & \\ & 2 & & 6 & & 6 & & 2 & \\ 1 & & 6 & & 10 & & 6 & & 1 \\ & 2 & & 6 & & 6 & & 2 & \\ & & 1 & & 2 & & 1 & & \end{array}$$

We can introduce secondary principal directions with the six-arrow symbol . This has the binary mask:

$$\begin{array}{ccccccccccc} & & & & 1 & & 1 & & & & \\ & 1 & & 2 & & 2 & & 2 & & 1 & \\ 1 & & 2 & & 4 & & 4 & & 2 & & 1 \\ & 2 & & 4 & & 4 & & 4 & & 2 & \\ 1 & & 2 & & 4 & & 4 & & 2 & & 1 \\ & 1 & & 2 & & 2 & & 2 & & 1 & \\ & & & & 1 & & 1 & & & & \end{array}$$

Arrows	z-transform	Mask	Continuity
$\rightarrow\rightarrow$	$3\left(\frac{(1+z+z^2)}{3}\right)^2$	$\frac{1}{3}[1,2,3,2,1]$	$C0$
$\rightarrow\rightarrow\rightarrow$	$3\left(\frac{(1+z+z^2)}{3}\right)^3$	$\frac{1}{9}[1,3,6,7,6,3,1]$	$C1$
$\rightarrow\rightarrow\rightarrow\rightarrow$	$3\left(\frac{(1+z+z^2)}{3}\right)^4$	$\frac{1}{27}[1,4,10,16,19,16,10,4,1]$	$C2$

Fig. 9. The linear, quadratic, and cubic ternary univariate box-spline subdivision schemes.

with four stencils, one for the vertex and three rotational variants for the three edges:

```
  2 2          1 2 1
 2 4 2          4 4
  2 2          1 2 1
```

However, a binary scheme *cannot* be factorised into two $\sqrt{2}$ steps, because there is no way to construct a triangular $\sqrt{2}$ subdivision scheme [14, 15]. Indeed, the longer arrows are $\sqrt{3}$ the length of the shorter arrows, rather than $\sqrt{2}$. To get a factorisable version, you must construct a ternary scheme, which can be factorised into the convolution of two $\sqrt{3}$ steps.

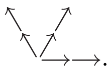
6 Ternary and Higher Arities

The arrow notation extends to box-spline schemes of higher arities. For example, the ternary arrow $\rightarrow$, which looks identical to the binary arrow, corresponds to $(1+z+z^2)/3$. This means that there are three possible ways in which the arrow can be used: no translation, a translation of one unit, and a translation of two units. Compare this with the binary arrow, which corresponds to $(1+z)/2$, where we can interpret it as either translation of one unit or no translation, which is equivalent to either using the arrow or not using it. So long as we remember that the geometric interpretation of the ternary arrow is somewhat different, we can proceed as for the binary case. In general, the n-ary arrow corresponds to $(1-z^n)/(n(1-z))$ which is $(1+z+\cdots+z^{n-1})/n$.

6.1 Ternary Univariate Schemes

As for the binary univariate schemes, we can list the possible ternary univariate box-spline schemes (Fig. 9). Remember that each of these schemes has three stencils, obtained by taking every third element from the mask. For example, the cubic ternary scheme has the three stencils $\frac{1}{27}[1,16,10]$, $\frac{1}{27}[4,19,4]$, and $\frac{1}{27}[10,16,1]$

6.2 Ternary Bivariate Schemes

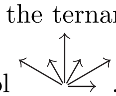
All of the derivations which work for the binary schemes also work for the ternary schemes. For example, we can easily derive the ternary mask for the arrow symbol . This produces the ternary version of Loop subdivision [16] which has the mask:

1 2 3 2 1
2 6 10 10 6 2
3 10 20 24 20 10 3
2 10 24 36 36 24 10 2
1 6 20 36 45 36 20 6 1
2 10 24 36 36 24 10 2
3 10 20 24 20 10 3
2 6 10 10 6 2
1 2 3 2 1

Selecting every third entry on every third line generates the stencils shown below, along with the five rotations of the second stencil and the one rotation of the third stencil:

6 6	2 10	3 24 3
6 45 6	1 36 20	24 24
6 6	2 10	3

These stencils are for new vertices at the vertex, a third of the way along an edge, and face centre respectively.

We can also generate the ternary version of the six-arrow, six-direction box spline, with arrow symbol . This has the mask:

1 1 1
1 2 3 3 2 1
1 2 4 6 7 6 4 2 1
1 3 6 9 11 11 9 6 3 1
1 3 7 11 15 16 15 11 7 3 1
2 6 11 16 19 19 16 11 6 2
1 4 9 15 19 21 19 15 9 4 1
2 6 11 16 19 19 16 11 6 2
1 3 7 11 15 16 15 11 7 3 1
1 3 6 9 11 11 9 6 3 1
1 2 4 6 7 6 4 2 1
1 2 3 3 2 1
1 1 1

This *can* be factorised into two $\sqrt{3}$ steps, which are much simpler to evaluate.

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & 1 & & 1 & \\
1 & & 2 & & 1 \\
 & 2 & & 2 & \\
1 & & 3 & & 1 \\
 & 2 & & 2 & \\
1 & & 2 & & 1 \\
 & 1 & & 1 & \\
 & & 1 & &
\end{array}
\quad * \quad
\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & 1 & & 2 & & 2 & & 1 & \\
1 & & 2 & & 3 & & 2 & & 1 \\
 & 1 & & 2 & & 2 & & 1 & \\
 & & 1 & & 1 & & 1 & &
\end{array}
$$

The stencils which can be derived from these $\sqrt{3}$ masks are:

$$
\begin{array}{ccccc}
 & 1 & & 1 & \\
1 & & 3 & & 1 \\
 & 1 & & 1 &
\end{array}
\qquad\qquad
\begin{array}{ccccc}
1 & & 2 & & 1 \\
 & 2 & & 2 & \\
 & & 1 & &
\end{array}
$$

for new vertices at the vertex and face centre respectively. Note that this is *not* the Kobbelt $\sqrt{3}$ scheme [17] and that the Kobbelt scheme is not a box-spline scheme.

7 Summary

The four different methods of deriving subdivision masks all have their benefits. The arrow notation is useful in that it is a graphical, rather than strictly mathematical, representation and in that it allows us to read off the continuity of the box-spline scheme directly. It also allows us to enumerate all possible schemes easily by forming all possible combinations of arrows. This paper has shown all possible symmetric binary quadrilateral schemes which have up to eight arrows, and all possible symmetric binary triangular schemes which have up to six arrows.

There are interesting small projects that could be tackled, arising from this work, each involving investigating the generalisation of a box-spline scheme or schemes to the extraordinary cases, edges and creases. They are:

1. an investigation of the family of dual quadrilateral binary box-spline schemes illustrated in Fig. 8;
2. an investigation of the $\sqrt{2}$ scheme described in Sect. 4, including consideration of the effect of arrows which are longer than the shortest primary and secondary arrows; and
3. an investigation of the $\sqrt{3}$ scheme described at the end of Sect. 6.

References

1. Dyn, N.: Analysis of convergence and smoothness by the formalism of Laurent polynomials. In Iske, A., Quak, E., Floater, M.S., eds.: Tutorials on Multiresolution in Geometric Modelling. Springer 51–68

2. Sabin, M.A., Augsdörfer, U.H., Dodgson, N.A.: Artifacts in box-spline surfaces. In Martin, R., Bez, H., Sabin, M., eds.: Mathematics of Surfaces XI. Volume 3604 of Lecture Notes in Computer Science., Springer (2005) 350–363
3. Sabin, M., Barthe, L.: Artifacts in recursive subdivision schemes. In Cohen, A., Merrien, J.L., Schumaker, L.L., eds.: Curve and Surface Fitting: Saint-Malo 2002, Nashboro Press (2003) 353–362
4. Peters, J., Shiue, L.J.: Combining 4- and 3-direction subdivision. ACM Trans. Graph. **23**(4) (2004) 980–1003
5. Peters, J., Reif, U.: The simplest subdivision scheme for smoothing polyhedra. ACM Trans. Graph. **16**(4) (1997) 420–431
6. Velho, L.: Quasi 4-8 subdivision. Computer Aided Geometric Design **18**(4) (2001) 345–357
7. Doo, D.: A subdivision algorithm for smoothing down irregularly shaped polyhedrons. In: Proceedings on Interactive Techniques in Computer Aided Design. (1978) 157–165 available online from `www.idi.ntnu.no/~fredrior/files/Doo%201978%20Subdivision%20algorithm.pdf` (Jan 2009).
8. Doo, D., Sabin, M.A.: Behaviour of recursive division surfaces near extraordinary points. Computer-Aided Design **10**(6) (1978) 356–360
9. Qu, R.: Recursive subdivision algorithms for curve and surface design. PhD thesis, Brunel University (1990)
10. Zorin, D., Schrder, P.: A unified framework for primal/dual quadrilateral subdivision schemes. Computer Aided Geometric Design **18**(5) (2001) 429–454
11. Dyn, N.: Subdivision schemes in computer-aided geometric design. In Light, W., ed.: Advances in Numerical Analysis, Vol. 2, Clarendon Press (1992) 36–104
12. Hassan, M.F., Ivrissimtzis, I.P., Dodgson, N.A., Sabin, M.A.: An interpolating 4-point C2 ternary stationary subdivision scheme. Computer Aided Geometric Design **19**(1) (2002) 1–18
13. Loop, C.T.: Smooth subdivision surfaces based on triangles. Master's thesis, University of Utah, Department of Mathematics (1987)
14. Ivrissimtzis, I.P., Dodgson, N.A., Sabin, M.A.: A generative classification of mesh refinement rules with lattice transformations. Computer Aided Geometric Design **21**(1) (2004) 99–109
15. Dodgson, N.A.: An heuristic analysis of the classification of bivariate subdivision schemes. In Martin, R.R., Bez, H.E., Sabin, M.A., eds.: Mathematics of Surfaces XI. Volume 3604 of Lecture Notes in Computer Science., Springer (2005) 161–183
16. Loop, C.: Smooth ternary subdivision of triangle meshes. In Cohen, A., Merrien, J.L., Schumaker, L.L., eds.: Curve and Surface Fitting: Saint-Malo 2002, Nashboro Press (2003) 295–302
17. Kobbelt, L.: $\sqrt{3}$ subdivision. In: Proc. ACM SIGGRAPH. (2000) 103–112