This is a preprint of a paper published in:
Proceedings of SMI2002: International Conference on Shape Modelling and Applications, 17-22 May 2002, Banff, Alberta, IEEE press, pp. 29-34, ISBN 0-7695-1546-0

Recursive subdivision and hypergeometric functions

I.P.Ivrissimtzis*, N.A.Dodgson†
University of Cambridge
Computer Laboratory

M.A.Sabin‡
Numerical Geometry Ltd

July 24, 2001

Abstract

We describe a method for efficient calculation of coefficients for subdivision schemes. We work on the unit sphere and we express the z-coordinate of all the existing points as power series in the variable $\cos\theta$. Any linear combination of them is also a power series in $\cos\theta$ and, by solving a linear system, we determine the linear combination that will give the smoothest interpolation of the sphere at a particular point. This way we are able to find constructively some optimal coefficients for the subdivision scheme.

Keywords: subdivision; Chebyshev polynomials; hypergeometric functions.

1 Introduction

In [3] a constructive method was proposed for the calculation of the coefficients of a subdivision scheme, based on geometric considerations rather than on solving an optimisation problem. We worked on the sphere S with centre the origin O and radius 1. We identified the parameter space with the plane $z = 0$, allowing us to describe the exact position of the points with a function

$$f_z : \mathbf{R}^2 \to \mathbf{R} \tag{1}$$

giving their z coordinates.

On the sphere S, except of the Euclidean metric inherited by its embedding in $\mathbf{R}^3$, we can define the spherical metric, where the distance between two points is equal to the Euclidean angle θ between the vectors from the centre of the sphere to these two points. In this paper we will always assume S to

* *ipi20@cam.ac.uk*
† *nad@cl.cam.ac.uk*
‡ *malcolm@geometry.demon.co.uk*

be equipped with this spherical metric. The z-coordinate of any point P of S is given by $\cos\theta$ where θ is the distance between P and the top of the sphere (0,0,1).

In [3] we studied subdivision supposing that the existing points lie on S and are locally symetrically arranged around $(0, 0, 1)$. Exploiting the symmetry, we grouped them in subsets, so that for every such set of points the sum of their inverses images in the parameter space gives the origin O. That allowed us to deal separately with the z-coordinate. We searched for linear combinations of these set of points, with sum of weights 1, lying on S. Then we used the coefficients of these linear combinations to describe subdivision schemes that were optimal from the point of view of the spherical geometry we employed.

In one of the examples we studied in [3] we showed that we can use the Chebyshev polynomials of the first kind and their first derivatives to implement efficiently these methods to the study of the 4-point schemes $(b, 2)$ introduced in [2]. Here we will generalise these ideas using higher order derivatives of the Chebyshev polynomials and their generalisation the hypergeometric functions.

We will again group the points in subsets so that the sum of the inverse images in the parameter space is the origin. In the simplest case we suppose that all the points in a subset have the same distance from (0,0,1). We write all the distances as multiples of a basic distance θ, that is

$$c_1\theta, c_2\theta, \dots c_n\theta \tag{2}$$

All the z-coordinates are given by

$$\cos c_1\theta, \cos c_2\theta, \dots \cos c_n\theta \tag{3}$$

and we will write all the above functions as power series in $\cos\theta$.

We suppose that the newly inserted point of the subdivision scheme is the image of the point (0,0) of the parameter space. We express it as a linear combination of the existing points and by linearity its z-coordinate will have the form of a linear combination

$$\alpha_1 \cos c_1\theta + \alpha_2 \cos c_2\theta + \cdots + \alpha_n \cos c_n\theta \tag{4}$$

which also can be written as a power series $\mathcal{P}(\cos\theta)$. To find coefficients for a stationary scheme we will study the limit case $\theta \to 0$, that is, $\cos\theta \to 1$. So, we will study the power series $\mathcal{P}$ at 1. We will require $\mathcal{P}(1) = 1$, which means that the z-coordinate of the newly inserted point is 1 and so it lies on S, actually it is the point (0,0,1), that is, the top of S. Considering $\mathcal{P}$ as a local approximation of S at (0,0,1), we will require that its first k derivatives are equal to 0 at $\mathcal{P}(1)$, where k is a large as possible. That is, the approximation will be the smoothest possible.

In the next sections we will see how we can calculate the coefficients α_i so that these requirements are met. We will also see a modification of this method with grouping of points with different distances from (0,0,1).

2 The univariate case. Chebyshev polynomials

The univariate case is special in the sense that usually, that is in all the known uniform schemes, all the distances $c_1\theta, c_2\theta, \ldots, c_n\theta$ can be expressed as integer multiples of a basic distance θ. In particular, in the binary $(2, N)$ schemes which were introduced in [2], if we suppose that the existing points are equally spaced on the unit circle with centre (0,0), their distances from the new inserted point are

$$\theta,\ 3\theta,\ 5\theta,\ \ldots\ ,\ (2n-1)\theta \tag{5}$$

Working as in the bivariate case, we identify the parameter space $\mathbf{R}$ with the line $y = 0$, that is the x-axis, suppose that the new point is the image of the point 0 of the parameter space and that the existing points are symmetrically arranged around it. Then the linear combination (4), giving the y-coordinate of the new point, becomes

$$\alpha_1 \cos\theta + \ \alpha_2 \cos 3\theta + \ \alpha_3 \cos 5\theta + \ \cdots \ + \ \alpha_n \cos(2n-1)\theta \tag{6}$$

The function $\cos n\theta$ can be expressed as a polynomial of variable $\cos\theta$ with the use of the Chebyshev polynomial of the first kind

$$\cos n\theta = T_n(\cos\theta) \tag{7}$$

Writing $\cos\theta = x$ we get the linear combination of Chebyshev polynomials

$$\mathcal{P}(x) = \alpha_1 T_1(x) + \alpha_2 T_3(x) + \alpha_3 T_5(x) + \cdots + \alpha_n T_{2n-1}(x) \tag{8}$$

As in the bivariate case we require $\mathcal{P}(1) = 1$, so that the new point also lies on the unit circle, and we use the rest $n-1$ degrees of freedom to force the first $n-1$ derivatives of $\mathcal{P}$ to be 0 at 1.

So, the system of equations that will give the smoothest possible approximation of the unit circle at the point (0,1) is

$$\begin{array}{lllll}
\alpha_1 T_1(1) & +\alpha_2 T_3(1) & + \ \cdots & +\alpha_n T_{2n-1}(1) & = 1 \\
\alpha_1 T_1'(1) & +\alpha_2 T_3'(1) & + \ \cdots & +\alpha_n T_{2n-1}'(1) & = 0 \\
\vdots & \vdots & \vdots \quad \vdots & & \vdots \\
\alpha_1 T_1^{(n-1)}(1) & +\alpha_2 T_3^{(n-1)}(1) & + \ \cdots & +\alpha_n T_{2n-1}^{(n-1)}(1) & = 0
\end{array}$$

To calculate the values of the Chebyshev polynomials and their derivatives at 1, we will consider a generalisation of them, that is, the Gegenbauer or ultraspherical polynomials. For a real number $a > -\frac{1}{2}$ the Gegenbauer polynomials

$C_n^a,\ n=0,1,2,\ldots$ are the family of orthogonal polynomials corresponding to the weight function

$$w(x) = (1-x^2)^{a-\frac{1}{2}} \tag{9}$$

see [1]. We have

$$T_n(x) = \frac{n}{2}C_n^0(x) \tag{10}$$

For the first derivative we have the formula

$$T_n'(x) = nC_{n-1}^1(x) \tag{11}$$

while for higher derivatives we can use the formula

$$\frac{d}{dx}C_n^a(x) = 2aC_{n-1}^{a+1}(x) \tag{12}$$

Combining equations (11) and (12) we get

$$T_n^{(k)}(x) = n2^{k-1}(k-1)!C_{n-k}^k(x), \quad k=1,2,\ldots n \tag{13}$$

Also, see [1], we have

$$C_n^a(1) = \binom{n+2a-1}{n},\ a \neq 0, \quad C_n^0(1) = \frac{2}{n},\ n \neq 0, \quad C_0^0(1) = 1 \tag{14}$$

and so,

$$T_n(1) = 1, \quad T_n^{(k)}(1) = n2^{k-1}(k-1)!\binom{n+k-1}{n-k} \tag{15}$$

We notice that any equation of the system, except the first, can be simplified by extracting a common factor $2^{k-1}(k-1)!$, and the system becomes

$$\begin{array}{llllll}
\alpha_1 & +\alpha_2 & +\cdots & +\alpha_{n-1} & +\alpha_n & =1 \\
\alpha_1\binom{1}{0} & +\alpha_2 3\binom{3}{2} & +\cdots & +\alpha_{n-1}(2n-3)\binom{2n-3}{2n-4} & +\alpha_n(2n-1)\binom{2n-1}{2n-2} & =0 \\
0 & +\alpha_2 3\binom{4}{1} & +\cdots & +\alpha_{n-1}(2n-2)\binom{2n-2}{2n-5} & +\alpha_n(2n-1)\binom{2n}{2n-3} & =0 \\
0 & +\alpha_2 3\binom{5}{0} & +\cdots & +\alpha_{n-1}(2n-1)\binom{2n-1}{2n-6} & +\alpha_n(2n-1)\binom{2n+1}{2n-4} & =0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & +0 & +\cdots & +\alpha_{n-1}(2n-3)\binom{3n-5}{n-2} & +\alpha_n(2n-1)\binom{3n-3}{n} & =0
\end{array}$$

The treatment of the more general $2N$-point k-ary subdivision, where $k-1$ new points are inserted between any two consecutive existing points, is a little more complicated. To calculate the coefficients for the rth new point we notice that the distances from the existing points are

$$m_1\theta, m_2\theta, \dots, m_{2n}\theta \tag{16}$$

with

$$\begin{aligned} m_1 = (N-1)k + r,\ m_2 = (N-2)k + r,\ \dots\ , m_n = r \\ m_{n+1} = k - r,\ \dots\ , m_{2n-1} = (N-1)k - r,\ m_{2n} = Nk - r \end{aligned} \tag{17}$$

Here we cannot consider a linear combination of the above points and work as in the binary case, because nothing guarantees that the x coordinate of that linear combination will be 0. Instead, we will pair the points P_j and P_{2n-j+1} with distances m_j and m_{2n-j+1} corresponding, for $j = 1, \dots, n$, and we will work with their linear combination

$$\frac{m_{2n-j+1}}{m_j + m_{2n-j+1}} P_j + \frac{m_j}{m_j + m_{2n-j+1}} P_{2n-j+1} \tag{18}$$

which is a point with x coordinate equal to

$$\frac{m_{2n-j+1}}{m_j + m_{2n-j+1}} \sin m_j\theta + \frac{m_j}{m_j + m_{2n-j+1}} \sin m_{2n-j+1}\theta \tag{19}$$

which has a limit equal to 0 for $\theta \to 0$, as well as its first two derivatives.

The functions giving the y-coordinate are

$$\frac{m_{2n-j+1}}{m_j + m_{2n-j+1}} \cos m_j\theta + \frac{m_j}{m_j + m_{2n-j+1}} \cos m_{2n-j+1}\theta \qquad j = 1, \dots, n \tag{20}$$

Finally, working similarly to the binary case and using equation (15), we get the linear system

$$\begin{array}{llllll} \alpha_1 & +\alpha_2 & + \cdots & +\alpha_n & = 1 \\ \alpha_1 a_{21} & +\alpha_2 a_{22} & + \cdots & +\alpha_n a_{2n} & = 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_1 a_{n1} & +\alpha_2 a_{n2} & + \cdots & +\alpha_n a_{nn} & = 1 \end{array}$$

with

$$
\begin{gathered}
a_{ij} = \frac{m_{2n-j+1}}{m_j + m_{2n-j+1}} m_j \binom{m_j + i - 2}{m_j - i + 1} + \\
+ \frac{m_j}{m_j + m_{2n-j+1}} m_{2n-j+1} \binom{m_{2n+j-1} + i - 2}{m_{2n+j-1} - i + 1} \\
i = 2, \ldots, n \quad j = 1, \ldots, n
\end{gathered} \tag{21}
$$

under the convention that if the binomial symbol is not well defined we consider it 0. This case arises when the order of the derivative is higher than the degree of the polynomial.

In the next section we will see that in the bivariate case, a more general approach will lead to an even simpler system equivalent to the above.

3 The bivariate case. Hypergeometric functions

In the bivariate case all the distances are again expressed in terms of multiples, but not necessarily integer, of a basic distance θ. That is, we have to deal with functions of the form $\cos c\theta$ where c is a real number. In that case the expansion of these functions as power series of variable $\cos\theta$ is given with the use of hypergeometric functions.

We have, see [1]

$$
\cos 2a\theta = F(-a, a; \frac{1}{2}; \sin^2\theta) \tag{22}
$$

which is a power series of variable $\sin^2\theta$, see equation (25) below. For the sake of completeness of exposition, in the appendix at the end of the paper, we will carry over the calculations transforming (22) into an expression of $\cos a\theta$ as a power series of variable $\cos\theta$. But here we will write $\cos a\theta$ as a power series of variable $(1-\cos\theta)$. The main benefit is that we will have to evaluate that power series and its derivatives at 0 rather than at 1, and so will work with finitely many terms of it rather than infinite many.

Writing equation (22) with $\frac{\theta}{2}$ instead of θ, gives

$$
\cos a\theta = F(-a, a; \frac{1}{2}; \sin^2\frac{\theta}{2}) \tag{23}
$$

giving,

$$
\cos a\theta = F(-a, a; \frac{1}{2}; \frac{1}{2}(1-\cos\theta)) \tag{24}
$$

We also have,

$$
F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \tag{25}
$$

where $(a)_k$ is the Pochhammer symbol defined by

$$
(a)_0 = 1, \quad (a)_k = a(a+1)(a+2)\cdots(a+k-1), \; k = 1, 2, \ldots \tag{26}
$$

see [4]. That gives,

$$\cos a\theta = P_a(1-\cos\theta) = \sum_{n=0}^{\infty} \frac{(-a)_n(a)_n}{(\frac{1}{2})_n} \frac{(1-\cos\theta)^n}{2^n n!} \tag{27}$$

Now, writing $x = \cos\theta$ the linear combination (4) becomes

$$\alpha_1 P_{c_1}(1-x) + \alpha_2 P_{c_2}(1-x) + \cdots + \alpha_n P_{c_n}(1-x) \tag{28}$$

and with the same geometric and analytic reasoning as in the case of Chebychev polynomials we will calculate α_i's such that the above linear combination has value 1 at 1, while its first $n-1$ derivatives have value 0 at that point. That is, we have to solve the system

$$\begin{array}{lllll}
\alpha_1 P_{c_1}(0) & +\alpha_2 P_{c_2}(0) & + \cdots & +\alpha_n P_{c_n}(0) & = 1 \\
\alpha_1 P'_{c_1}(0) & +\alpha_2 P'_{c_2}(0) & + \cdots & +\alpha_n P'_{c_n}(0) & = 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_1 P^{(n-1)}_{c_1}(0) & +\alpha_2 P^{(n-1)}_{c_2}(0) & + \cdots & +\alpha_n P^{(n-1)}_{c_n}(0) & = 0
\end{array}$$

The above system is satisfied if and only if the constant coefficient of the polynomial

$$\alpha_1 P_{c_1}(x) + \alpha_2 P_{c_2}(x) + \cdots + \alpha_n P_{c_n}(x) \tag{29}$$

is equal to 1 and the coefficients of $x, \ldots, x^{n-1}$ are equal to 0.
So, using (26) the system becomes

$$\begin{array}{lllll}
\alpha_1 \frac{(-c_1)_0(c_1)_0}{(\frac{1}{2})_0 2^0 0!} & +\alpha_2 \frac{(-c_2)_0(c_2)_0}{(\frac{1}{2})_0 2^0 0!} & + \cdots & +\alpha_n \frac{(-c_n)_0(c_n)_0}{(\frac{1}{2})_0 2^0 0!} & = 1 \\
\alpha_1 \frac{(-c_1)_1(c_1)_1}{(\frac{1}{2})_1 2^1 1!} & +\alpha_2 \frac{(-c_2)_1(c_2)_1}{(\frac{1}{2})_1 2^1 1!} & + \cdots & +\alpha_n \frac{(-c_n)_1(c_n)_1}{(\frac{1}{2})_1 2^1 1!} & = 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_1 \frac{(-c_1)_{n-1}(c_1)_{n-1}}{(\frac{1}{2})_{n-1} 2^{n-1}(n-1)!} & +\alpha_2 \frac{(-c_2)_{n-1}(c_2)_{n-1}}{(\frac{1}{2})_{n-1} 2^{n-1}(n-1)!} & + \cdots & +\alpha_n \frac{(-c_n)_{n-1}(c_n)_{n-1}}{(\frac{1}{2})_{n-1} 2^{n-1}(n-1)!} & = 0
\end{array}$$

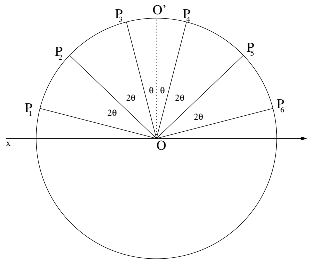

Figure 1: Six equally distanced points symmetrically arranged around O' the top of the unit circle.

giving,

$$
\begin{array}{lllll}
\alpha_1 & +\alpha_2 & +\cdots & +\alpha_n & =1 \\
\alpha_1(-c_1)_1(c_1)_1 & +\alpha_2(-c_2)_1(c_2)_1 & +\cdots & +\alpha_n(-c_n)_1(c_n)_1 & =0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_1(-c_1)_{n-1}(c_1)_{n-1} & +\alpha_2(-c_2)_{n-1}(c_2)_{n-1} & +\cdots & +\alpha_n(-c_n)_{n-1}(c_n)_{n-1} & =0
\end{array}
$$

4 Examples

We will give two univariate and one bivariate examples to illustrate the use of the methods described above.

4.1 A 6-point scheme

Suppose that the six points

$$P_1, P_2, \ldots, P_6 \tag{30}$$

are equally distanced and symmetrically arranged around the top $O' = (0,1)$ of the unit circle. See fig.[1]
Their distances from O' are $\theta, 3\theta, 5\theta$ and by section 2 we have to solve the system

$$
\begin{array}{cccc}
\alpha_1 & +\alpha_2 & +\alpha_3 & =1 \\
\alpha_1\binom{1}{0} & +\alpha_2 3\binom{3}{2} & +\alpha_3 5\binom{5}{4} & =0 \\
 & \alpha_2 3\binom{4}{1} & +\alpha_3 5\binom{6}{3} & =0
\end{array}
$$

giving,

$$\begin{array}{llll} \alpha_1 & +\alpha_2 & +\alpha_3 & = 1 \\ \alpha_1 & +9\alpha_2 & +25\alpha_3 & = 0 \\ & 12\alpha_2 & +100\alpha_3 & = 0 \end{array}$$

which has the solution,

$$\alpha_1 = \frac{75}{64} \quad \alpha_2 = \frac{-25}{128} \quad \alpha_3 = \frac{3}{128} \tag{31}$$

Dividing each coefficient by 2, which is the number of points corresponding to it, we find the coefficients $\frac{75}{128}, \frac{-25}{256}, \frac{3}{256}$ of the subdivision scheme. That is, we found the $(2,3)$ scheme described in [2].

4.2 A 4-point ternary scheme

In section 2 we saw that sometimes because of a certain lack of symmetry we have to group together points with different y coordinate. To illustrate that we will calculate the coefficients for a uniform 4-point ternary univariate scheme with the method of section 2.

Using the terminology of section 2, for $r = 1$ we have

$$m_1 = 4 \qquad m_2 = 1 \qquad m_3 = 2 \qquad m_4 = 5 \tag{32}$$

The system of section 2 becomes

$$\begin{array}{lll} \alpha_1 & +\alpha_2 & = 1 \\ \alpha_1(\frac{5}{9}4\binom{4}{3} + \frac{4}{9}5\binom{5}{4}) & +\alpha_2(\frac{2}{3}1\binom{1}{0} + \frac{1}{3}2\binom{2}{1}) & = 0 \end{array}$$

giving,

$$\begin{array}{lll} \alpha_1 & +\alpha_2 & = 1 \\ 20\alpha_1 & +2\alpha_2 & = 1 \end{array}$$

which has the solution

$$\alpha_1 = -\frac{1}{9} \quad \alpha_2 = \frac{10}{9} \tag{33}$$

and the coefficients of the scheme can be found using (18). We have

$$-\frac{1}{9}(\frac{4}{9}P_1 + \frac{5}{9}P_4), \qquad \frac{10}{9}(\frac{2}{3}P_2 + \frac{1}{3}P_3) \tag{34}$$

that is,

$$-\frac{4}{81}P_1, \ \frac{20}{27}P_2, \ \frac{10}{27}P_3, \ -\frac{4}{81}P_4 \tag{35}$$

Because of a particular symmetry of uniform ternary schemes, for $r = 2$ we just get a permutation of the above ratio of distances, and we do not need to solve a second linear system. We also notice that the scheme we found is the (3,2) scheme described in [2].

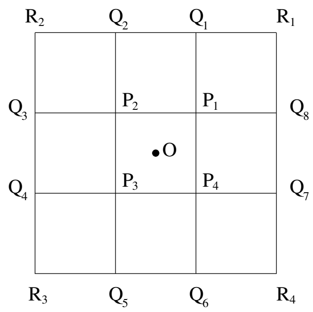

Figure 2: The mask of a 16-point $\sqrt{2}$-scheme.

4.3 A $\sqrt{2}$-scheme for regular grids

In [3] we described a subdivision scheme for regular grids belonging to the $\sqrt{2}$ family. In each step of that scheme a new point is inserted in the centre of each face, calculated as linear combination of its nearest 16-points. See figure [2]. Then every new vertex is connected with its 4 nearest neighbours, while the original edges are removed causing a 45 degree rotation of the grid.

The 16 points are grouped according to their distances from the new point O as

$$\{P_1, P_2, P_3, P_4\},\ \{Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_7, Q_8\},\ \{R_1, R_2, R_3, R_4\} \tag{36}$$

We take the distance between P_i and O as the basic distance θ. Unlike the univariate case it is not straightforward what the other distances should be, because it depends on the embedding of fig.[2] on the sphere, which cannot be as regular as its embedding in the Euclidean plane. Because we are interested in the limit case $\theta \to 0$, that is, in the embedding of fig.[2] in a small neighbourhood around the top of the sphere, and because in that case the ratio of the distances on the sphere tends to the ratio of the distances on the Euclidean plane, we will suppose that

$$OP_i = \theta,\ OQ_i = \sqrt{5}\theta,\ OR_i = 3\theta \tag{37}$$

So, the system becomes

$$\begin{array}{cccc} \alpha_1 & +\alpha_2 & +\alpha_3 & = 1 \\ (-1)1\alpha_1 & (-\sqrt{5})\sqrt{5}\alpha_2 & (-3)3\alpha_3 & = 0 \\ & (-\sqrt{5})(-\sqrt{5}+1)\sqrt{5}(\sqrt{5}+1)\alpha_2 & (-3)(-2)3 \cdot 4\alpha_3 & = 0 \end{array}$$

giving,

$$\begin{array}{cccc} \alpha_1 & +\alpha_2 & +\alpha_3 & = 1 \\ -\alpha_1 & -5\alpha_2 & -9\alpha_3 & = 0 \\ & 20\alpha_2 & +72\alpha_3 & = 0 \end{array}$$

which has the solution

$$\alpha_1 = \frac{45}{32} \quad \alpha_2 = \frac{-9}{16} \quad \alpha_3 = \frac{5}{32} \tag{38}$$

Finally, we divide each coefficient with the number of points corresponding to it, and we find the coefficients $\frac{45}{128}, \frac{-9}{128}, \frac{5}{128}$ for the points P_i, Q_i, R_i, respectively. That is, we found the same coefficients as in [3] where we worked with a different method.

We can notice that if all the distances are rational or square roots of rationals, as it is always the case on a regular grid, then all the coefficients of the system are rational. Indeed, the number $(c_i)_n(-c_i)_n$ is the product of numbers of the form

$$(c_i + k)(-c_i + k) = -c_i^2 + k^2, \quad k = 0, 1, \ldots, n-1 \tag{39}$$

which is rational if c_i is rational or the square root of a rational. In that case the solutions of the system and thus the coefficients of the subdivision scheme are also rational.

5 Conclusion - Further work

We developed a method to calculate efficiently coefficients for subdivision schemes, taking into consideration geometric and analytic aspects of subdivision. We showed that the study of subdivision can be facilitated with mathematical tools such as the Chebyshev polynomials and the hypergeometric functions.

We conjecture that the univariate schemes generated by the method described in section 2, are the same as the schemes (b, N) in [2]. In [3], in a slihgtly different context, there was a proof for the case $N = 2$.

6 Appendix

Equation (22) with $\frac{a}{2}$ instead of a gives

$$\cos a\theta = F(-\frac{a}{2}, \frac{a}{2}; \frac{1}{2}; \sin^2\theta) \tag{40}$$

We use the linear transformation of the variable, see [1]

$$\begin{gathered} F(a, b; c; z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a, b; a+b-c+1; 1-z) + \\ +(1-z)^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} F(c-a, c-b; c-a-b+1; 1-z) \end{gathered} \tag{41}$$

and (40) becomes

$$\begin{gathered} \cos a\theta = \frac{\Gamma^2(\frac{1}{2})}{\Gamma(\frac{1+a}{2})\Gamma(\frac{1-a}{2})} F(-\frac{a}{2}, \frac{a}{2}; \frac{1}{2}; \cos^2\theta) + \\ + \cos\theta \frac{\Gamma(\frac{1}{2})\Gamma(-\frac{1}{2})}{\Gamma(-\frac{a}{2})\Gamma(\frac{a}{2})} F(\frac{1+a}{2}, \frac{1-a}{2}; \frac{3}{2}; \cos^2\theta) \end{gathered} \tag{42}$$

Using (25) we get,

$$\cos a\theta = P_a(\cos\theta) = \frac{\Gamma^2(\frac{1}{2})}{\Gamma(\frac{1+a}{2})\Gamma(\frac{1-a}{2})}\sum_{n=0}^{\infty}\frac{(-\frac{a}{2})_n(\frac{a}{2})_n}{(\frac{1}{2})_n}\frac{\cos^{2n}\theta}{n!} + \\ + \frac{\Gamma(\frac{1}{2})\Gamma(-\frac{1}{2})}{\Gamma(-\frac{a}{2})\Gamma(\frac{a}{2})}\sum_{n=0}^{\infty}\frac{(\frac{1+a}{2})_n(\frac{1-a}{2})_n}{(\frac{3}{2})_n}\frac{\cos^{2n+1}\theta}{n!} \tag{43}$$

which expresses $\cos a\theta$ as a power series of variable $\cos\theta$. The first part of the sum gives the even powers of $\cos\theta$ and the second part the odd.

To simplify the formula we can use the formulae, see [5],

$$\Gamma(\frac{1}{2}) = \sqrt{\pi} \quad \text{and} \quad \Gamma(-\frac{1}{2}) = -2\sqrt{\pi} \tag{44}$$

and the reflection formulae

$$\Gamma(\frac{1}{2}+z)\Gamma(\frac{1}{2}-z) = \frac{\pi}{\cos\pi z} \tag{45}$$

giving,

$$\Gamma^2(\frac{1}{2}) = \pi \quad , \quad \Gamma(\frac{1}{2})\Gamma(-\frac{1}{2}) = -2\pi \quad , \quad \Gamma(\frac{1+a}{2})\Gamma(\frac{1-a}{2}) = \frac{\pi}{\cos\frac{\pi a}{2}} \tag{46}$$

and

$$\Gamma(-\frac{a}{2})\Gamma(\frac{a}{2}) = \frac{\Gamma(-\frac{a}{2}+1)}{-\frac{a}{2}}\Gamma(\frac{a}{2}) = -\frac{2}{a}\frac{\pi}{cos\frac{(a-1)\pi}{2}} = -\frac{2}{a}\frac{\pi}{cos\frac{(1-a)\pi}{2}} \tag{47}$$

So, equation (43) becomes

$$\cos a\theta = P_a(\cos\theta) = \cos\frac{\pi a}{2}\sum_{n=0}^{\infty}\frac{(-\frac{a}{2})_n(\frac{a}{2})_n}{(\frac{1}{2})_n}\frac{\cos^{2n}\theta}{n!} + \\ + a\cos\frac{(1-a)\pi}{2}\sum_{n=0}^{\infty}\frac{(\frac{1+a}{2})_n(\frac{1-a}{2})_n}{(\frac{3}{2})_n}\frac{\cos^{2n+1}\theta}{n!} \tag{48}$$

References

[1] M.Abramowitz & I.Stegun. Handbook of mathematical functions with formulas, graphs, and mathematical tables. *Wiley-Interscience,* (1972).

[2] G.Deslauriers & S.Dubuc. Symmetric Iterative Interpolation Processes. *Constructive Approximation,* **5**, pp.49-68, (1989).

[3] I.Ivrissimtzis & N.Dodgson & M.Hassan & M.Sabin. On the geometry of recursive subdivision. *(submitted for publication)*, (2001).

[4] J.Seaborn. Hypergeometric functions and their applications. *TAM Springer-Verlag,* (1991).

[5] D.Zwillinger (Editor-in-chief). Standard Mathematical Tables and formulae. *CRC Press,* (1996).