

Quadratic Interpolation for Image Resampling

Neil A. Dodgson

Abstract—Nearest-neighbour, linear, and various cubic interpolation functions are frequently used in image resampling. Quadratic functions have been disregarded, largely because they have been thought to introduce phase distortions. This is shown not to be the case, and a family of quadratic functions is derived. The interpolating member of this family has visual quality close to that of the Catmull-Rom cubic, yet requires only sixty percent of the computation time.

Keywords—Image resampling, quadratic interpolation, cubic interpolation

I. INTRODUCTION

IMAGE resampling is used in a wide variety of image manipulation tasks including image scaling, image registration, image warping, and photo-mosaicing. Resampling is often divided into two sub-processes: reconstruction and sampling. The former creates a continuous function from the discrete image data, and the latter samples this to create a new, resampled, image. Reconstruction of a piecewise continuous function from discrete data is most often taken to be a linear combination of the input data and a reconstruction kernel. For unit-spaced samples this is:

$$f(x) = \sum_{i=-\infty}^{+\infty} f_i h(x-i)$$

where the f_i are the sample values and $h(s)$ is the reconstruction kernel.

Many different reconstruction kernels have been proposed and analysed [1]–[9]. In most cases the reconstruction kernel is a major factor in the quality of the final image. Speed of processing is frequently also an issue, and so a trade-off between time and quality may have to be made.

Piecewise local polynomials are used extensively for reconstruction in image resampling applications because they are simple, quick to evaluate, and easy to implement. Examples are nearest-neighbour, linear, and Catmull-Rom cubic interpolation. The first two are extremely simple to evaluate, but give poor quality visual results. The latter takes longer to evaluate but generally results in a good image.

Detailed investigation of degree 0, 1, and 3 piecewise local polynomials has been carried out [1], [3]–[6], [9]–[11]. However, the degree 2 (quadratic) polynomials have been largely disregarded. This correspondence discusses why they have been ignored, derives a set of potentially useful quadratics, and analyses these by comparison to well-known degree 0, 1, and 3 polynomials.

Note that these functions do not necessarily pass through the input data points. We use the convention that a function that passes through all the input data points is an *interpolant*, while one that does not is an *approximant* [9].

N. A. Dodgson is with the Computer Laboratory, University of Cambridge, Pembroke Street, Cambridge CB2 3QG, U.K. E-mail: nad@cl.cam.ac.uk.

II. PIECEWISE LOCAL POLYNOMIALS

The piecewise local polynomials have been used extensively in image resampling applications. They are simple, easily evaluated functions. In one dimension they have kernels of the general form:

$$h(s) = \begin{cases} \sum_{j=0}^{n-1} k_{i,j} s^j, & \begin{cases} i \leq s < i+1, \\ i = \lfloor s \rfloor, \\ m \text{ even,} \\ |s| < \frac{m}{2} \end{cases} \\ \sum_{j=0}^{n-1} k_{i,j} s^j, & \begin{cases} i - \frac{1}{2} \leq s < i + \frac{1}{2}, \\ i = \lfloor s + \frac{1}{2} \rfloor, \\ m \text{ odd,} \\ |s| < \frac{m}{2} \end{cases} \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

When convolved with the data points this produces a piecewise equation of degree $n-1$ where each polynomial piece depends on the nearest m data points. The $k_{i,j}$ are the $m \times n$ co-efficients of the equation. They are usually determined by applying constraints to the form of the reconstructed function. Normally $m = n$, although there are a few exceptions to this rule (as in Maeland's two point cubic interpolant, for which $m = 2$ and $n = 4$ [6]).

This equation can be trivially extended to two dimensions by using the same one dimensional kernel along each axis of the co-ordinate system [1]. In this correspondence we will present one-dimensional versions for clarity. References to visual quality will, of course, be to the two-dimensional versions.

III. ZERO, FIRST, AND THIRD-DEGREE POLYNOMIALS

The simplest, non-trivial, example of a piecewise, local polynomial is the zero-degree member of the family ($m = n = 1$), nearest-neighbour interpolation:

$$h(s) = \begin{cases} 1, & |s| \leq \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

In image resampling this is a very fast operation, but it produces an image with blocky artifacts. An example of a function reconstructed by this method can be seen in Figure 1(a), while its kernel is shown in Figure 2(a).

The only useful first-degree member of the family ($m = n = 2$) is the familiar linear interpolation:

$$h(s) = \begin{cases} 1 - |s|, & |s| \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

In resampling work this usually gives a better result than nearest-neighbour, for an extra computational cost. Unfortunately it also tends to blur the final image. Figure 1(b)

shows a set of points reconstructed by linear interpolation, and Figure 2(b) shows the kernel.

Temporarily skipping over the quadratics (second degree) we come to the cubic members of the family ($m = n = 4$). These have been widely studied [1], [3]–[6], [8], [9]. Of particular note here is Mitchell and Netravali’s analysis of a two-parameter family of these cubics [1], against which we will be making some comparisons with the quadratic family of reconstructors. The two-parameter family of cubics have kernels of the form:

$$h(s) = \begin{cases} (2 - \frac{3}{2}B - C)|s|^3 \\ + (-3 + 2B + C)|s|^2 \\ + (1 - \frac{1}{3}B), & |s| \leq 1 \\ (-\frac{1}{6}B - C)|s|^3 \\ + (B + 5C)|s|^2 \\ + (-2B - 8C)|s| \\ + (\frac{4}{3}B + 4C), & 1 < |s| \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

where B and C are the two parameters.

Particular members of this family are of special note: the cubic $B = 1, C = 0$ is the approximating cubic B-spline (Figures 1(f) and 2(f)) which produces quite a blurry image, while $B = 0, C = \frac{1}{2}$ (Figures 1(e) and 2(e)) is the Catmull-Rom cubic which produces a good image.

IV. QUADRATIC RECONSTRUCTORS

Quadratics have been largely disregarded in image resampling. Two distinct reasons for this are given in the literature. The first, and more serious, objection to using quadratics is that their filters “are space-variant with phase distortion” [8]. Were this true, they would indeed be unsuitable for image resampling, as they would introduce distortions into the resampled image. Schafer and Rabiner [12] show that any quadratic *will* produce phase distortions *if* each quadratic piece starts and ends at the sample points. Making an alternative assumption, implicit in Equation 1, that each quadratic piece starts and ends halfway between sample points (i.e. each piece is based on the three *nearest* points) then, as will be seen, quadratics with linear phase can be derived that are suitable for image resampling. All reconstructors considered in this correspondence are space-invariant.

The second, more subjective, objection to using quadratics is that using three points for interpolation would result in two points on one side of the interpolated point and only one point on the other side [5]. Compare this with linear reconstruction, where there is one point either side, and cubic reconstruction, where there are two. The case for the quadratic can be restated by saying, as we did above, that the reconstructed point is calculated from the three *nearest* sample points, in the same way that linear interpolation is calculated from the two nearest, and cubic from the four nearest. The former statement produces a *perceived* preference for linear and cubic over quadratic, while the latter makes no distinction.

In our derivation of the quadratic we start with the general form (Equation 1 with $m = n = 3$):

$$h(s) = \begin{cases} as^2 + bs + c, & -\frac{3}{2} \leq s < -\frac{1}{2} \\ ds^2 + es + g, & -\frac{1}{2} \leq s \leq +\frac{1}{2} \\ us^2 + vs + w, & +\frac{1}{2} < s \leq +\frac{3}{2} \\ 0, & \text{otherwise} \end{cases} \quad (2)$$

To produce a useful reconstructor from this general form we need to apply restrictions to the equation. Following [1] and [9], we want the reconstructor to have linear phase (otherwise it will introduce phase distortions, which, as discussed above, are undesirable), C0-continuous (the eye being extremely sensitive to discontinuities in intensity [13]), and exhibit the property of ‘straightness’. This latter property is that, if the three control points for a given segment have the same value then the reconstructed segment should have that value for its entire length [9]. Mitchell and Netravali [1] describe this latter condition as “designing the problem of sample-frequency ripple out of the reconstructor”, while Parker *et al* [5] describe it as “ensuring that the DC amplification is unity”.

Enforcing these three eminently sensible criteria reduces Equation 2 to one degree of freedom:

$$h(s) = \begin{cases} (-2r)|s|^2 + \frac{1}{2}(r+1), & |s| \leq \frac{1}{2} \\ (r)|s|^2 + (-2r - \frac{1}{2})|s| + \frac{3}{4}(r+1), & \frac{1}{2} < |s| \leq \frac{3}{2} \\ 0, & \text{otherwise} \end{cases} \quad (3)$$

All members of this one parameter family of quadratics have linear phase by virtue of the criteria applied in their derivation, proving the existence of piecewise quadratics which do not introduce phase distortions.

To remove the final degree of freedom, we can enforce either of two conditions, producing two specific reconstructors. The first condition is that the quadratic must interpolate the data points. This sets $r = 1$, reducing Equation 3 to:

$$h(s) = \begin{cases} -2|s|^2 + 1, & |s| \leq \frac{1}{2} \\ |s|^2 - \frac{5}{2}|s| + \frac{3}{2}, & \frac{1}{2} < |s| \leq \frac{3}{2} \\ 0, & \text{otherwise} \end{cases} \quad (4)$$

This quadratic’s kernel can be found in Figure 2(c), and an example reconstruction in Figure 1(c).

The second condition, which cannot be enforced at the same time as the first, because of the limited number of degrees of freedom available in Equation 3, is that the quadratic must be C1-continuous. This forces $r = \frac{1}{2}$, reducing Equation 3 to:

$$h(s) = \begin{cases} -|s|^2 + \frac{3}{4}, & |s| \leq \frac{1}{2} \\ \frac{1}{2}|s|^2 - \frac{3}{2}|s| + \frac{9}{8}, & \frac{1}{2} < |s| \leq \frac{3}{2} \\ 0, & \text{otherwise} \end{cases} \quad (5)$$

Figure 2(d) shows this quadratic’s kernel, and Figure 1(d) an example reconstruction.

We call these two functions the interpolating quadratic, and the approximating quadratic B-spline, respectively. Other values of r produce a blend of these two specific

quadratics, in a similar fashion to the way in which cubics falling along the line $2C + B = 1$ in Mitchell and Netravali's two-parameter family of cubics [1] are a blend of the Catmull-Rom interpolating cubic and the approximating cubic B-spline.

V. EVALUATION

A. Functional Evaluation

Most of the functional features of the family are criteria applied in their derivation. After these, perhaps the most important thing to note about the functional structure of the one parameter family is that they are constrained to pass through the midpoints of the segments joining each pair of adjacent data points. This constraint is a direct result of imposing the C0-continuity and straightness criteria on the form of the kernel.

The interpolating quadratic (Equation 4) produces a piecewise reconstruction where each parabolic piece is constrained to pass through a data point and the two adjacent midpoints. The piecewise curve is thus exactly defined by the only possible pieces that can pass through the controlling points. These pieces can be shown to be linear blends of two linear functions, as illustrated in Figure 3. This is analogous to Brewer and Anderson's derivation of the Catmull-Rom (Overhauser) cubic function as a linear blend of two interpolating parabolas [14]. The interpolating quadratic function is one of the general set of Catmull-Rom curves [15].

The approximating quadratic (Equation 5) is also constrained to pass through the midpoints but not through the sample points. It seems counter-intuitive to have a function which is constrained to pass through non-sample points and yet does not, in general, pass through the sample points themselves. However, this function is of good lineage, being the second degree B-spline.

B. Visual Evaluation

An important test of the visual quality of any reconstructor is visual inspection of the intensity surfaces that it produces. While a formal measure of quality is highly desirable, finding such a measure which is closely related to subjective quality has proved difficult [9], [13], [16].

Visually comparing the two quadratic reconstructors with linear interpolation, we find that the approximating quadratic B-spline produces a more blurry intensity surface than linear interpolation while the interpolating quadratic produces a less blurry result than the linear. Figure 4 shows an example image resampled (scaled by a factor of four) using each of the reconstructors.

If we compare these quadratics to similar cubic reconstructors we find the blurriness of the approximating quadratic lies between that of linear interpolation, and that of the excessively blurry approximating cubic B-spline (the approximating cubic B-spline should not be confused with the generally good results of the non-local interpolating cubic B-spline [6], [9], [17]). The surprising result is that the intensity surface produced by the quadratic interpolant

approaches the quality of that generated by the Catmull-Rom cubic, which is generally accepted as the best cubic interpolant [9]. The quadratic does exhibit a little more anisotropy than the Catmull-Rom but, apart from this, they are practically identical. A visual evaluation of the range of reconstructors provided by Equation 3 shows that a value around $r = 0.80$ provides the best trade off between the slight anisotropy of the interpolant and the blurriness of the approximant. This is similar to Mitchell and Netravali's *subjectively* best cubic [1] which is a sum of two-thirds Catmull-Rom and one-third approximating cubic B-spline, thus providing a trade off between the sometimes excessive sharpness of the Catmull-Rom and the blurriness of the B-spline.

C. Frequency Domain Analysis

A frequency domain analysis compares the reconstructors against the sinc function, the perfect reconstructor for band-limited signals [18], [19]. While this is accepted practice, it has been noted that it does not correlate well with subjective visual analysis [8], [9], [11]. Figures 5 and 6 show the frequency responses of various reconstructors. Only the positive frequency axis is shown as all of the functions are even.

Two things are of interest in the pass band: how soon the function starts to fall away, and how steeply it falls away. The closer to $\nu = 0.5$ and the steeper the fall off, the better [5]. Nearest-neighbour interpolation performs fairly well in the pass band but falls off very slowly. Linear interpolation has poor pass band performance with the quadratic and cubic approximating B-splines getting progressively worse, as could be expected from their progressively greater blurring effects. The Catmull-Rom cubic performs well, with the best pass band response and the fastest fall off. The interpolating quadratic's pass band response falls roughly halfway between the poor linear and the good Catmull-Rom.

In the stop band a better performance is indicated by a function which stays closest to zero. Nearest-neighbour thus has appalling performance, with the other B-splines (linear, quadratic, and cubic) getting progressively better. The approximating cubic B-spline has the best stop band performance of any function considered. The interpolating quadratic and Catmull-Rom cubic have similar stop band responses, the Catmull-Rom's being slightly better.

D. Summary of Evaluation

The best pass band performance of the six functions is shown by the Catmull-Rom cubic, followed by the interpolating quadratic. The best stop band performance is shown by the approximating cubic B-spline, followed by the Catmull-Rom. The approximating cubic B-spline has, however, appalling pass band performance. The interpolating quadratic has average stop band performance. The interpolating quadratic thus has a frequency response better than the linear but not as good as the Catmull-Rom cubic.

The surprising result of the visual analysis is that the

interpolating quadratic produces a visual result that is very nearly as good as the Catmull-Rom cubic's. The Catmull-Rom cubic or Mitchell and Netravali's subjectively best cubic [1] both produce slightly better visual results than this quadratic, and so would probably be used in preference in an application where quality is the overriding issue. In time-critical applications, however, the faster computation time of the quadratic over the cubic could offset the slight loss in quality that may result. Linear interpolation, while even faster, would represent a much greater loss of quality. A trial across nine two-dimensional resampling operations showed that a quadratic took 55–63% of the time of a cubic for similar visual quality, while a linear interpolation took 25–36% of the time of a cubic for significantly degraded visual quality.

VI. SUMMARY

We have shown that linear-phase, piecewise quadratics do exist and can be used for image resampling. We have derived two potentially useful quadratic functions. One of these (the interpolating quadratic) has visual quality approaching that of the Catmull-Rom cubic, but it requires only 60% of the computation time of the cubic. This could make it useful in applications where speed and quality must be traded off against one another.

Acknowledgements

Part of this work was funded by grants from the Cambridge Commonwealth Trust and the Committee of Vice-Chancellors and Principals of the UK.

REFERENCES

- [1] D. P. Mitchell and A. N. Netravali, "Reconstruction filters in computer graphics", *Comp. Graphics*, Vol. 22, No. 4, Aug, 1988, pp.221–288.
- [2] H. S. Hou and H. C. Andrews, "Cubic splines for image interpolation and digital filtering", *IEEE Trans. Acoust., Speech, Sig. Proc.*, Vol. 26, No. 6, Dec, 1978, pp.508–517.
- [3] R. G. Keys, "Cubic convolution interpolation for digital image processing", *IEEE Trans. Acoust., Speech, Sig. Proc.*, Vol. 29, No. 6, Dec, 1981, pp.1153–1160.
- [4] S. K. Park and R. A. Schowengerdt, "Image reconstruction by parametric cubic convolution", *Comp. Vision, Graphics and Image Proc.*, Vol. 23, No. 3, Sept, 1983, pp.258–272.
- [5] J. A. Parker, R. V. Kenyon and D. E. Troxel, "Comparison of interpolation methods for image resampling", *IEEE Trans. Med. Imag.*, Vol. 2, No. 1, Mar, 1983, pp.31–39.
- [6] E. Maeland, "On the comparison of the interpolation methods", *IEEE Trans. Med. Imag.*, Vol. 7, No. 3, Sept, 1988, pp.213–217.
- [7] P. S. Heckbert [1989], "Fundamentals of texture mapping and image warping", report no. UCB/CSD 89/516, Computer Science Division (EECS), University of California, Berkeley, California 94720, USA; June, 1989.
- [8] G. Wolberg, *Digital Image Warping*, IEEE Computer Society Press Monograph, IEEE Computer Society Press, 10662 Los Vaqueros Circle, P.O.Box 3013, Los Alamitos, CA 90720–1264, USA; 1990.
- [9] N. A. Dodgson, *Image Resampling*, Technical Report No. 261, Computer Laboratory, University of Cambridge, Pembroke Street, Cambridge CB2 3QG, U.K, Aug, 1992.
- [10] W. F. Schreiber and D. E. Troxel, "Transformation between continuous and discrete representations of images: a perceptual approach", *IEEE Trans. Pattern Anal. Mach. Intel.*, Vol. 7, No. 2, Mar, 1985, pp.178–186.
- [11] S. E. Reichenbach and S. K. Park, "Two-parameter cubic convolution for image reconstruction", *Proc. SPIE*, Vol. 1199, 1989, pp.833–840.
- [12] R. W. Schafer and L. R. Rabiner, "A digital signal processing approach to interpolation", *Proc. IEEE*, Vol. 61, No. 6, Jun, 1973, pp.692–702.
- [13] W. K Pratt, *Digital Image Processing*, Wiley, New York, USA, 1978.
- [14] J. A. Brewer and D. C. Anderson, "Visual interaction with Overhauser curves and surfaces", *Comp. Graphics*, Vol. 11, No. 2, Summer, 1977, pp.132–137.
- [15] E. Catmull and R. Rom, "A class of local interpolating splines", *Computer Aided Geometric Design*, R.E. Barnhill and R.F. Riesenfeld (eds.), Academic Press, New York, 1974, pp.317–326.
- [16] J. P. Oakley and M. J. Cunningham, "A function space model for digital image sampling and its application in image reconstruction", *Comp. Graphics and Image Proc.*, Vol. 49, No. 2, Feb, 1990, pp.171–197.
- [17] M. Unser, A. Aldroubi and M. Eden, "Fast B-spline transforms for continuous image representation and interpolation", *IEEE Trans. Pattern Anal. Mach. Intel.*, Vol. 13, No. 3, Mar, 1991, pp.277–285.
- [18] E. T. Whittaker, "On the functions which are represented by the expansions of the interpolation theory", *Proc. Royal Soc. of Edinburgh*, Vol. 35, Part 2, 1914–1915 session (issued separately July 13, 1915), pp.181–194.
- [19] C. E. Shannon, "Communication in the presence of noise", *Proc. IRE*, Vol. 37, No. 1, Jan, 1949, pp.10–21.

Figures for “Quadratic Interpolation for Image Resampling”

Neil A. Dodgson

June 14, 2001

This paper was published in the conventional manner, with the figures being sent to the publisher separately from the text. This document reproduces Figures 1–3, 5 and 6. Figure 4 is currently only available in the printed document:

“Quadratic interpolation in image resampling”, N. A. Dodgson, *IEEE Transactions on Image Processing* 6(9), Sep 1997, pp. 1322–1326.

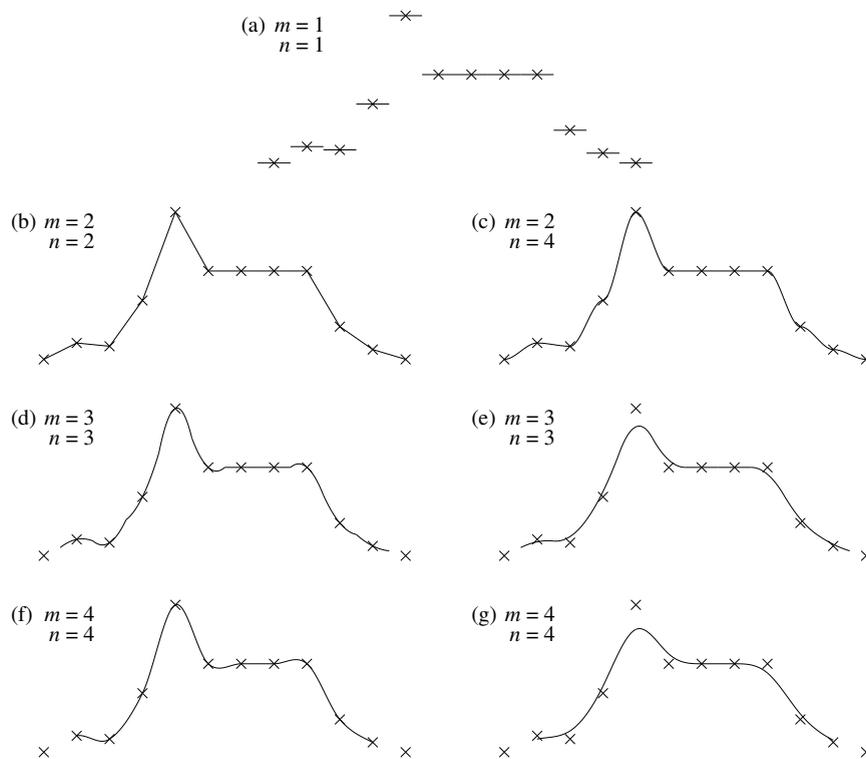


Figure 1: Example reconstructed functions. The input data points are represented by crosses. (a) nearest-neighbour, (b) linear, (c) interpolating quadratic, (d) approximating quadratic B-spline, (e) Catmull-Rom cubic, (f) approximating cubic B-spline.

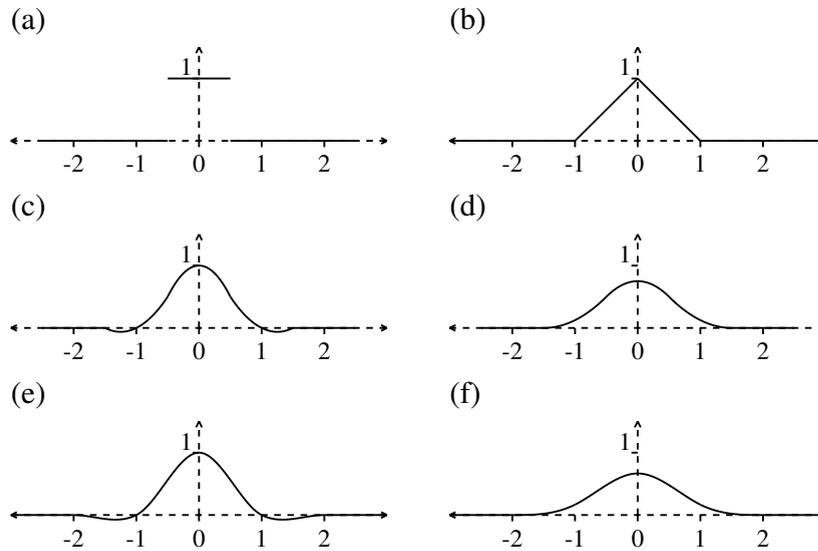


Figure 2: Kernels (also known as impulse responses): (a) nearest-neighbour, (b) linear, (c) interpolating quadratic, (d) approximating quadratic B-spline, (e) Catmull-Rom cubic, (f) approximating cubic B-spline.

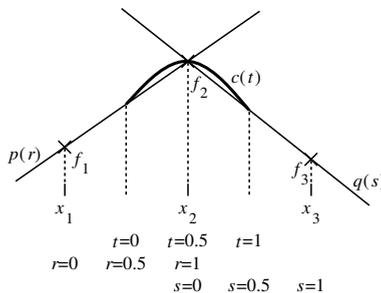


Figure 3: The quadratic pieces reconstructed by the interpolating quadratic kernel can be constructed as a linear blend of two linear functions. $p_{1,2}(t)$ and $p_{2,3}(t)$ are straight lines passing through the data points as shown. $q_{1,2,3}(t)$ is a linear blend of these, producing a quadratic piece which passes through the central data point, and the midpoints of the two lines. $q_{1,2,3}(t)$ is calculated from $p_{1,2}(t)$ and $p_{2,3}(t)$ as: $q_{1,2,3}(t) = (1 - t)p_{1,2}(t) + (t)p_{2,3}(t)$.

*This figure is
UNAVAILABLE
See printed paper for details.*

Figure 4: An image of Leominster Town Hall, enlarged by a factor of four using six different reconstructors: (a) nearest-neighbour, (b) linear, (c) interpolating quadratic, (d) approximating quadratic B-spline, (e) Catmull-Rom cubic, (f) approximating cubic B-spline.

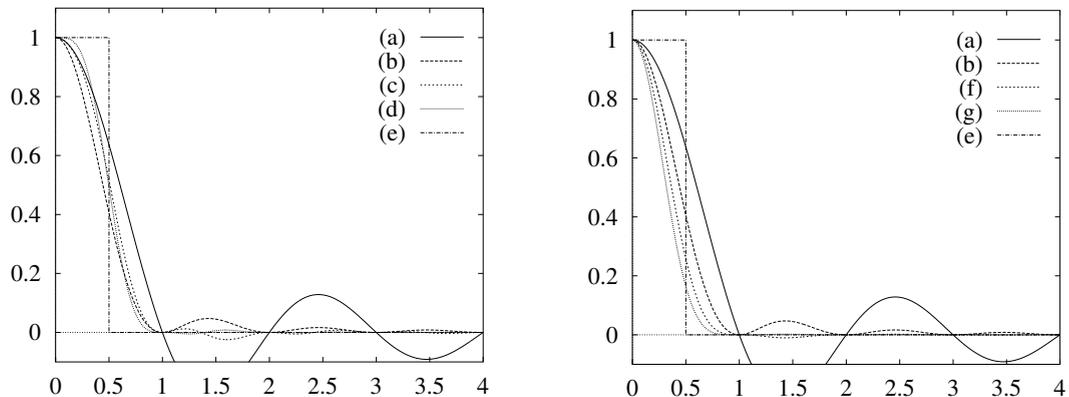


Figure 5: Frequency responses on a linear scale for evaluation of pass band performance. (a) nearest-neighbour, (b) linear, (c) interpolating quadratic, (d) Catmull-Rom cubic, (e) ideal (sinc) frequency response, (f) approximating quadratic B-spline, (g) approximating cubic B-spline.

The logarithmic scale graphs in Figure 6 are presented as printed in IEEE Transactions on Image Processing. The vertical scale is in decibels. The scale is incorrect in that it is scaled by a factor of $\ln(10)$: the values on the vertical axis therefore need to be divided by a factor of 2.3 to be correct.

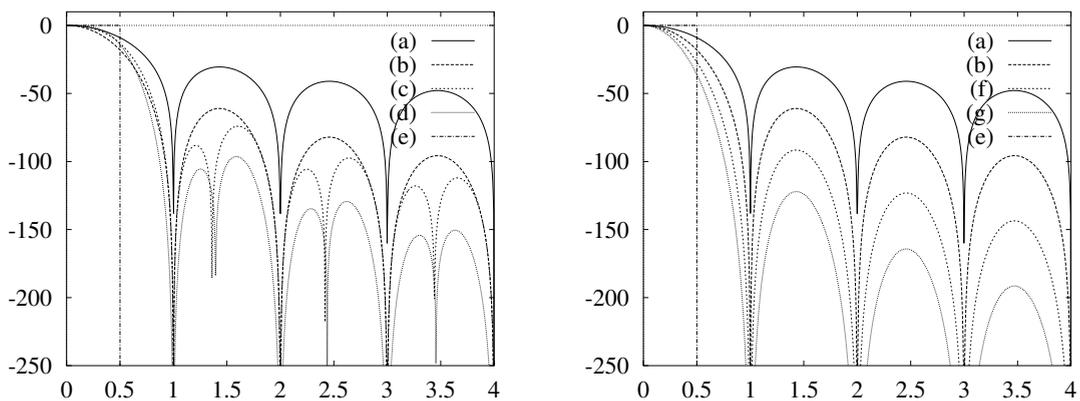


Figure 6: Frequency responses on a logarithmic (decibel) scale for evaluation of stop band performance. (a) nearest-neighbour, (b) linear, (c) interpolating quadratic, (d) Catmull-Rom cubic, (e) ideal (sinc) frequency response, (f) approximating quadratic B-spline, (g) approximating cubic B-spline.